

# Pascal's Triangle

## Teacher Notes and Answers

7 8 9 10 11 12



## Introduction

French mathematician Blaise Pascal [1623 – 1662] is not the creator of this famous triangle, the history of the triangle dates back through Chinese mathematics where it is still referred to as Yang Hui's triangle, and even further back to the Greeks [BC]. The title "Pascal's Triangle" was accredited posthumously by another French Mathematician Pierre Raymond de Montmort.

On the surface Pascal's triangle generates a set of numbers useful to probability and binomial expansion; however a whole treasure chest of patterns are hidden in this amazing triangle.



## Generating the Triangle

Watch the PowerPoint presentation on Pascal's triangle.

Watch to see how each successive number is produced on slide 1.

Teacher:

Use the Power Point slide show to support student learning.

Pascal's Triangle



Hidden Gems



## Question: 1

The first row in Pascal's triangle is Row zero (0) and contains a one (1) only. The PowerPoint animation reveals rows 0 through to 4. Draw these rows and the next three rows in Pascal's triangle.

Next three rows: Row 5: { 1, 5, 10, 10, 5, 1 }; Row 6: { 1, 6, 15, 20, 15, 6, 1 } and Row 7: { 1, 7, 21, 35, 35, 21, 7, 1 }.

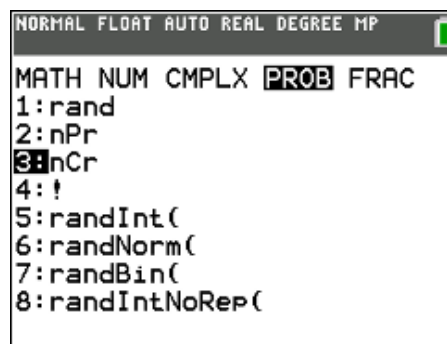
## Combinatorics and Polynomial Expansions

The probability menu contains the combinatorics command  $nCr$ ,

[ Math ] > Prob > nCr

Calculate each of the following:

${}^5C_0$ ,  ${}^5C_1$ ,  ${}^5C_2$  ...  ${}^5C_5$



## Question: 2

How do the calculations  ${}^5C_0$ ,  ${}^5C_1$ ,  ${}^5C_2$  ...  ${}^5C_5$  relate to the fifth row of Pascal's triangle?

**Answer:** These 'combinations' are the same as the fifth row of Pascal's triangle.

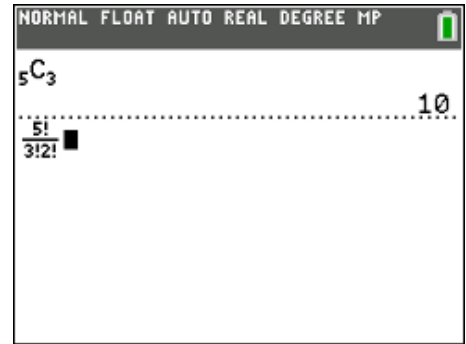
Compare the expression:

$$\frac{n!}{r!(n-r)!}$$

With those from the  ${}^n C_r$  calculations.

**Example:**  ${}^5 C_3 = 10$  as shown opposite has  $n = 5$  and  $r = 3$ .

The (!) sign is called 'factorial' and can be found in the probability menu immediately below the combinatorics command.



### Question: 3

Use the formula:  $\frac{n!}{r!(n-r)!}$  to calculate the 7<sup>th</sup> row of Pascal's triangle.

**Answer:** Row seven: { 1, 7, 21, 35, 35, 21, 7, 1 }

### Tip



To generate an entire row of Pascal's triangle, store the numbers in a list: { 0,1, 2, 3, 4, 5, 6, 7 } → L<sub>1</sub>. then use  ${}^7 C_{L_1}$  to see the entire row generated at once. This also works with the formula.

The sequence command can also be used:  $nCr(8,seq(x,x,0,8))$  to generate the 8<sup>th</sup> row.

### Question: 4

Write down the coefficients in each of the expansions below and locate these sets of numbers in Pascal's triangle?

- a)  $(x + y)^2$        $(x + y)^2 = x^2 + 2xy + y^2$  { 1, 2, 1 }
- b)  $(x + y)^3$        $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$  { 1, 3, 3, 1 }
- c)  $(x + y)^4$        $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$  { 1, 4, 6, 4, 1 }

Each set of numbers is located in the corresponding rows of Pascal's triangle.

### Question: 5

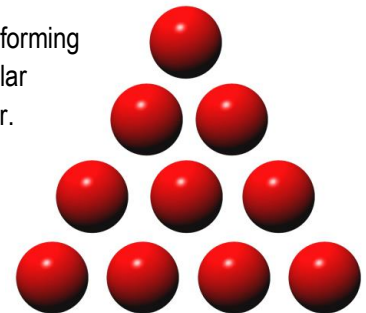
Use Pascal's triangle to identify the coefficients:  $a_0, a_1 \dots a_6$  in the expansion of  $(x + y)^6$  as shown below:

$$(x + y)^6 = a_0x^6 + a_1x^5y + a_2x^4y^2 + a_3x^3y^3 + a_4x^2y^4 + a_5xy^5 + a_6y^6$$

{ 1, 6, 15, 20, 14, 6, 1 } – The numbers are located in row 6 of Pascal's triangle.

## Triangular Numbers

Triangular numbers are essentially any number that can be represented by a dot pattern forming a triangle. Consider the triangle opposite; it contains 10 marbles (dots), so 10 is a triangular number. Remove the bottom row and you will have 6 marbles, so 6 is a triangular number. Add another row to the bottom of this triangle and it will contain 15 marbles, so 15 is a triangular number. The first four triangular numbers are: 1, 3, 6 and 10.



### Question: 6

Write down the first 10 triangular numbers.

**Answer:** { 1, 3, 6, 10, 15, 21, 28, 36, 45, 55 }

Watch the PowerPoint presentation on Pascal's triangle – triangular numbers to see how successive triangular numbers are created followed by a series of slides that provides some visual assistance to help establish a formula.



$n = 4$

Triangular Numbers



### Question: 7

After viewing the PowerPoint slides, including the 'establishing a formula slides', write a formula for Triangular numbers.

The 'establishing a formula' slides show a rectangle with dimensions  $n \times (n + 1)$ .

The triangular numbers make up half this rectangle. The formula is therefore:

$$T = \frac{n(n+1)}{2}$$

### Question: 8

The triangular numbers are in Pascal's triangle (shown opposite).

				1					
				1	1				
			1	2	1				
		1	3	3	1				
	1	4	6	4	1				
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		

- a) In which row is the fourth Triangular number?

The fourth triangular numbers is in row 5. The  $n^{\text{th}}$  triangular number is in row  $n + 1$

- b) Use combinatorics to write an expression for the  $n^{\text{th}}$  Triangular number and compare this formula to the one created in Question 7.

$$\begin{aligned} {}^{n+1}C_2 &= \frac{(n+1)!}{(n+1-2)!2!} \\ &= \frac{(n+1)(n+0)(n-1)(n-2)\dots}{2(n-1)(n-2)\dots} \\ &= \frac{n(n+1)}{2} \end{aligned}$$

To calculate  $n^{\text{th}}$  triangular number use  $(n + 1)^{\text{th}}$  row, element 2:

### Question: 9

Triangular numbers are the same as the sum of the first  $n$  natural numbers. The calculator can compute the sum of a set of numbers easily.

- a) Use sigma notation ( $\Sigma$ ) to calculate the sum of the first 3 numbers.  $(1 + 2 + 3) = \sum_{n=1}^3 n$  Sum = 6

**Hint:** Use [Alpha] [F2] to access the sigma notation.

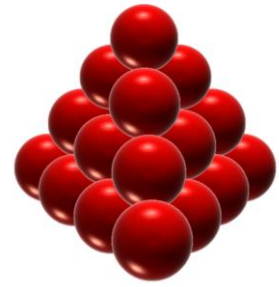
- b) Use the sigma notation to calculate the sum of the first 8 numbers.  $\sum_{n=1}^8 n = 36$

- c) Use the formula generated from the previous questions to check the answers produced here.

$$\frac{x(x+1)}{2} = \frac{10 \times 11}{2} = 55 \qquad \frac{x(x+1)}{2} = \frac{3 \times 4}{2} = 6$$

## Tetrahedral Numbers

The Tetrahedral numbers are a three dimensional version of the triangular numbers. A tetrahedron is a triangular pyramid. In the pyramid shown opposite the top row contains one ball. The second row contains three providing a total of four balls. The third row contains six balls for a total of ten.



The first three tetrahedral numbers are therefore: { 1, 4, 10 }

### Question: 10

Tetrahedral numbers are a figurate number representing the total number of objects required to make a tetrahedron.

- a) The first four tetrahedral numbers are: { 1, 4, 10, 20 }. Where do the tetrahedral numbers appear in Pascal's triangle? **In the (next) diagonal from the triangular numbers.**
- b) The tetrahedral numbers can be computed as the sum of the triangular numbers. The fifth tetrahedral number

can be computed as:  $(1 + 3 + 6 + 10 + 15) = \sum_{n=1}^5 \frac{n(n+1)}{2}$ .

Use the sigma notation to determine the 6<sup>th</sup> tetrahedral number.  $\sum_{n=1}^6 \frac{n(n+1)}{2} = 56$

- d) Use sigma notation ( $\Sigma$ ) to help determine a formula for the tetrahedral numbers.

**Hint:** Use the formula computed for triangular numbers in the sum and plot them on a graph.

**Answer:** Students should generate a table of values and may use regression, simultaneous equations,

matrices, difference tables or other strategies to determine the equation:  $\sum_{n=1}^x \frac{n(n+1)}{2} = \frac{x(x+1)(x+2)}{6}$

```
HISTORY
{0,1,2,3,4,5}→L1
{0 1 2 3 4 5}
∑x=05 (  $\frac{x*(x+1)}{2}$  )
35
{0,1,4,10,20,35}→L2
{0 1 4 10 20 35}
CubicReg L1,L2
Done
```

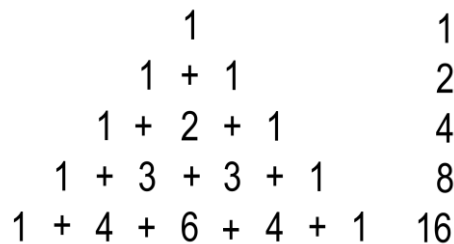
```
NORMAL FLOAT AUTO REAL DEGREE MP
CubicReg
y=ax3+bx2+cx+d
a=0.1666666667
b=0.5
c=0.3333333333
d=0
```

- e) Given the location of the tetrahedral numbers in Pascal's triangle, determine the formula for the tetrahedral numbers using combinatorics.

$$\begin{aligned} {}^{n+2}C_3 &= \frac{(n+2)!}{(n+2-3)!3!} \\ &= \frac{(n+2)(n+1)(n+0)(n-1)(n-2)\dots}{6(n-1)(n-2)\dots} \\ &= \frac{n(n+1)(n+2)}{6} \end{aligned}$$

### Powerful Numbers

Pascal's triangle is created by adding pairs of numbers to create elements in the next row, but what happens if you add all the numbers in each row? Rows zero through to four are shown opposite.



**Question: 11**

Generate the values in the 10<sup>th</sup> row of Pascal's triangle, calculate the sum and confirm that it fits the pattern.

$$\sum_{x=0}^{10} {}^{10}C_x = 1024. \quad \text{Also } 2^{10} = 1024$$

**Question: 12**

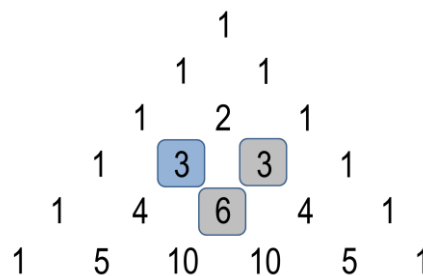
Given the relationship between the coefficients of  $(x + y)^n$  and Pascal's triangle, explain why the sum of each row produces this set of numbers.

$$(x + y)^{10} = x^{10} + 10x^9y + 45x^8y^2 \dots + 10xy^9 + y^{10}$$

Substitute  $x = 1$  and  $y = 1$  produces:  $(1 + 1)^{10} = (1)^{10} + 10(1)^9(1) + 45(1)^8(1)^2 \dots + 10(1)(1)^9 + (1)^{10}$   
 $(2)^{10} = 1 + 10 + 45 \dots + 10 + 1$

### Square Numbers

Each row of Pascal's triangle is generated by repeated and systematic addition. The result of this repeated addition leads to many multiplicative patterns. Another relationship in this amazing triangle exists between the second diagonal (natural numbers) and third diagonal (triangular numbers). If you square the number in the 'natural numbers' diagonal it is equal to the sum of the two adjacent triangular numbers (shown opposite).



$$3^2 = 6 + 3$$

The same is true for the next number in the natural numbers diagonal:

$$4^2 = 10 + 6$$

**Question: 13**

Show that this relationship holds for the next 3 numbers in the natural numbers diagonal.

Fifth Row (fifth natural number):  $5^2 = 10 + 15.$

Sixth Row (sixth natural number):  $6^2 = 15 + 21.$

Seventh Row (seventh natural number):  $7^2 = 21 + 28.$

**Question: 14**

Show this relationship holds for all numbers in the 'natural numbers diagonal' and the 'triangular numbers' diagonal.

Using combinatorics:

LHS

$$\begin{aligned} \binom{n}{1}^2 &= \left( \frac{n!}{(n-1)!} \right)^2 \\ &= \left( \frac{(n+0)(n-1)(n-2)\dots}{(n-1)(n-2)\dots} \right)^2 \\ &= n^2 \end{aligned}$$

RHS

$$\begin{aligned} {}^n C_2 &= \frac{n!}{2(n-2)!} & {}^{n+1} C_2 &= \frac{(n+1)!}{2(n-1)!} \\ &= \frac{n(n-1)(n-2)(n-3)\dots}{2(n-2)\dots} + & &= \frac{(n+1)n(n-1)(n-2)(n-3)\dots}{2(n-1)(n-2)\dots} \\ &= \frac{n(n-1)}{2} & &= \frac{n(n+1)}{2} \end{aligned}$$

Adding the two quantities:

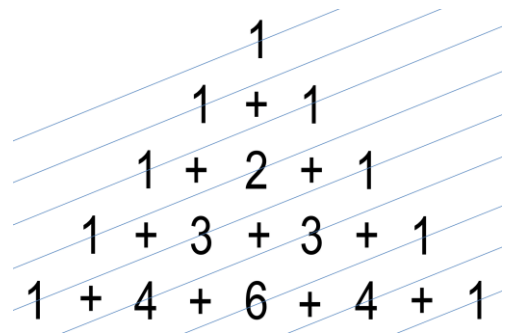
$$\begin{aligned} \frac{n(n-1)}{2} + \frac{n(n+1)}{2} &= \frac{n^2 - n + n^2 + n}{2} \\ &= n^2 \end{aligned}$$

Shown: LHS = RHS

**Shallow Diagonals**

Shallow diagonals can cross multiple rows in Pascal's triangle. The triangle must be drawn accurately (as shown) to identify all the numbers that lie on the shallow diagonals.

Diagonal:	Numbers:	Sum:
1	1	1
2	1	1
3	1, 2	3
4	1, 3, 1	5
5	1, 4, 3	8

**Question: 15**

Determine the numbers that lie on the next two diagonals, their corresponding sum and the pattern that is being formed with the sums of these diagonals.

Next two diagonals:  $\{1, 6, 5, 1\}$   $\therefore 1 + 6 + 5 + 1 = 13$  and  $\{4, 10, 6, 1\} = 4 + 10 + 6 + 1 = 21$

The sequence is therefore: 1, 1, 3, 5, 8, 13, 21 ... Fibonacci Sequence.

## Exhaustive Products

Some patterns in Pascal's triangle are easier to find and prove than others. This pattern is one of the most amazing hidden gems in Pascal's triangle. The rule for this pattern is to find the product of the numbers in row  $n$ , and multiply this by the product of the numbers in row  $n + 2$ , then, divide the result by the product squared for the numbers in row  $n + 1$ . This rule can be written as:

$$\frac{\prod_{r=0}^{n-1} \binom{n-1}{r} \times \prod_{r=0}^{n+1} \binom{n+1}{r}}{\left(\prod_{r=0}^n \binom{n}{r}\right)^2}$$

For example, suppose  $n = 5$ .

$$\text{Row 4} = \{1, 4, 6, 4, 1\} \quad \prod_{r=0}^{n-1} \binom{n-1}{r} = \prod_{r=0}^4 \binom{4}{r} = 1 \times 4 \times 6 \times 4 \times 1 = 96$$

$$\text{Row 5} = \{1, 5, 10, 10, 5, 1\} \quad \prod_{r=0}^n \binom{n}{r} = \prod_{r=0}^5 \binom{5}{r} = 1 \times 5 \times 10 \times 10 \times 5 \times 1 = 25,000$$

$$\text{Row 6} = \{1, 6, 15, 20, 15, 6, 1\} \quad \prod_{r=0}^{n+1} \binom{n+1}{r} = \prod_{r=0}^6 \binom{6}{r} = 1 \times 6 \times 15 \times 20 \times 15 \times 6 \times 1 = 162,000$$

$$\frac{96 \times 162,000}{25,000^2} \approx \frac{7776}{3125} \approx 2.48832$$

### Tip



The product command is in the List menu and also the catalogue.

The syntax for the product command is: `prod( list, [start], [end])`. Start and End are optional.

### Question: 16

Determine the value of the products and resulting approximation when  $n = 10$ .

$$L_1 = \text{seq}(x, x, 0, 10)$$

$$\text{prod}({}^{10}C_{L_1}) =$$

Calculator screen show shows how these calculations can be done with reasonable efficiency.

HISTORY	
	3.24060912E16
<code>seq(X,X,0,9,1)→L1</code>	{0 1 2 3 4 5 6 7 8 9}
<code>Prod({}^9C_{L1})</code>	1.175952237E13
<code>seq(X,X,0,11,1)→L1</code>	{0 1 2 3 4 5 6 7 8 9 10 11}
<code>Prod({}^{11}C_{L1})</code>	2.31627686E20

Overall result can be obtained by copying and pasting each result into the final equation  $\approx 2.593742459$

### Question: 17

The values in this product get very big, very quickly. The maximum value the calculator can compute is for the 67<sup>th</sup>, 68<sup>th</sup> and 69<sup>th</sup> rows. It can be shown however that:

$$\frac{\prod_{r=0}^{n-1} \binom{n-1}{r} \times \prod_{r=0}^{n+1} \binom{n+1}{r}}{\left(\prod_{r=0}^n \binom{n}{r}\right)^2} = \left(1 + \frac{1}{n}\right)^n$$

- a) Compare the result from this algebraic equivalence by substituting  $n = 10$ .

Answers are very close: 2.59374246 (Calculator rounds result)

- b) It is possible to substitute larger values into this algebraic equivalence. What is the ratio when  $n = 1000$ .  
**Note:** This is equivalent to finding the product of the terms in the 999<sup>th</sup> row of Pascal's triangle and multiplying by the product of the terms in the 1001<sup>th</sup> row and dividing by the square of the product of the terms in the 1000<sup>th</sup> row! **When  $n = 1000$ , the ratio is approximately 2.7169**
- c) What transcendental number do you think this might be approaching?

**Student answers may vary, however given the very slow convergence from 10 through to 1000, it is unlikely the number will exceed 3, therefore the most common transcendental number close to this is  $e$  (Euler's number).**

## Triangles within Triangles

Modular arithmetic is sometimes called 'clock arithmetic'. What time will it be in 25 hours from now? The answer is simply "one hour past the current time." The same is true for 49 hours from now and also for 73 hours. We ignore the number of days, the multiples of 24 and focus on the 'remainder'. The calculator is capable of performing modular arithmetic. For the examples given above:  $\text{Mod}(25,24) = 1$ ;  $\text{Mod}(49,24) = 1$  and  $\text{Mod}(73,24) = 1$ . The first number in the Mod command is the quotient and the second number is the divisor, the answer is the 'remainder'.

### Question: 18

Explore what happens if  $\text{Mod}(n,2)$  is applied to every element in Pascal's triangle? You will need to create many lines to see the pattern.

**Hint:** Try colouring regions containing 1's in one colour and 0's in another.

**Answer:** The more rows of Pascal's triangle that are included, the more obvious will be the result. Regions of odd numbers (and even numbers) occur within triangles, even more impressive is that this triangle arrangement is also known as a Sierpinski Triangle.

### Teacher Notes:

So, if your students are not yet impressed by some of the amazing gems that can be found in Pascal's triangle, send them off looking for more! There are more. For example the CATALAN numbers also appear, amongst others.