Taylor Made Polynomials

Teachers Teaching with Technology™
Professional Development from Texas Instruments

Answers

7 8 9 10 **11** **12**

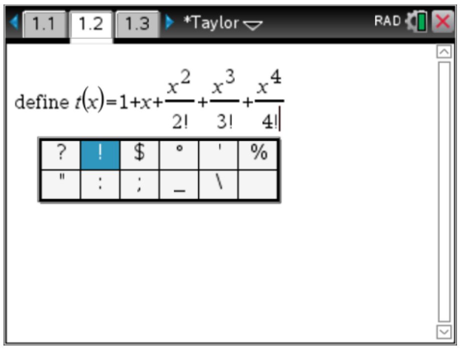
TI-Nspire | Investigation | Student | 45 min

Introduction

Brook Taylor (1685 – 1731) was an English Mathematician, he is acknowledged as formerly introducing the Taylor Series which built on the work of Scottish Mathematician James Gregory (1638 – 1675). The Taylor Series is the respresentation of a function as an infinite sum of terms. A Taylor polynomial is a finite number of terms that can be used to approximate the function. In this activity we will use Taylor Polynomials, but understand that the same ideas can be applied to the series.

Instructions

Open the TI-Nspire file: **Taylor Made**



Navigate to page 1.2 and define the following polynomial:

$$t(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

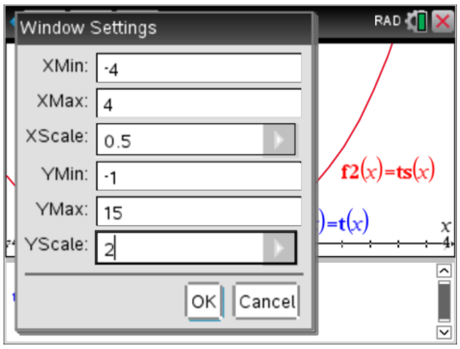
Navigate to page 1.3 to see a graph of the polynomial: $t(x)$

Question: 1.

Describe the shape of the graph looking through the Window settings provided on Page 1.3.

Over the domain shown, the graph resembles an exponential function. Note that even for this very abbreviated form t(1) ≈ 2.71.

The polynomial function (below) has been defined:



$$ts(x) = \sum_{n=0}^{m} \frac{x^n}{n!}$$

Change the window settings in the **Graph** application to match the settings shown then graph this function: $ts(x)$.

This new function contains a parameter (m). The slider value has been set to match: $t(x)$. Increase the value of m using the slider.

Question: 2.

Explain how the polynomial definition for $ts(x)$ works. (Use example values of m)

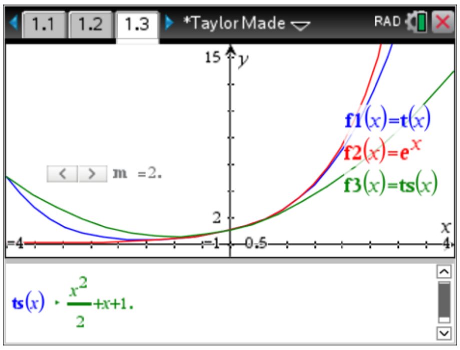
The value of *n* changes through {0, 1, 2, 3, 4} and are summed. So the first term is: $\frac{x^0}{0!} = 1$ since 0! is defined as 1. The second term is $\frac{x^1}{1!} = x$ and the third $\frac{x^2}{2!}$ …

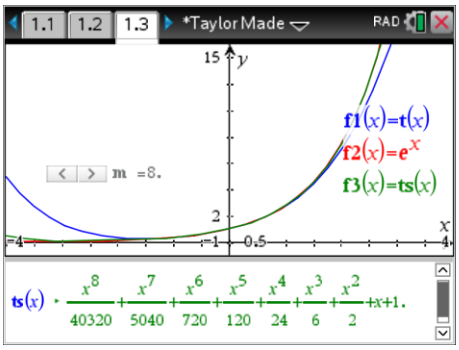
Author: P. Fox

Question: 3.

Successively increase the value for m (slider) and observe the changes to the graph. What function is this Taylor Polynomial modelling?

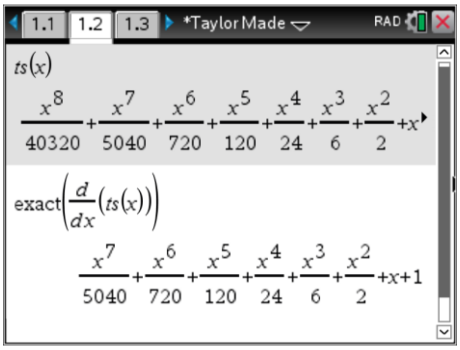
The graph becomes closer and closer to $f(x)=e^x$. Students may also choose to graph the two functions.





Question: 4.

Determine the derivative of this polynomial and explain how this supports your previous answer.



The derivative can be easily computed by hand or on the calculator. Notice that the two polynomials are almost identical. If the power of $ts(x)$ is allowed to increase then it is only the highest power that changes, all other terms will be the same. The function that is the same as its derivative is: $f(x)=e^x$

Modulus and Argument

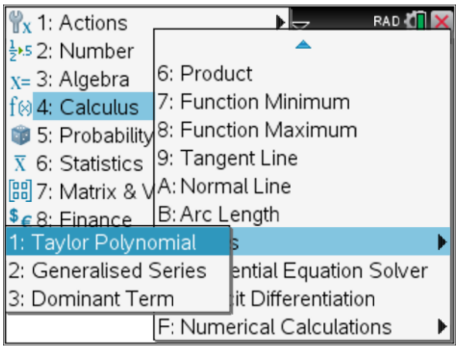
Navigate to page 2.1.

To generate a Taylor Polynomial use the calculus menu:

Calculus > **Series** > **Taylor Polynomial**

The syntax for the command is as follows:

Taylor(Expression, Variable, Order)



Question: 5.

Generate the following Taylor Polynomials:

i. Taylor(e^x , x, 8) $\quad 1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\dfrac{x^5}{5!}+\dfrac{x^6}{6!}+\dfrac{x^7}{7!}+\dfrac{x^8}{8!}$

ii. Taylor(cos(x), x, 8) $\quad 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\dfrac{x^8}{8!}$

Author: P. Fox

iii. Taylor(sin(x), x, 8) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}$

iv. Taylor(cos(x) + sin(x), x, 8) $1+x-\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}-\frac{x^6}{6!}-\frac{x^7}{7!}+\frac{x^8}{8!}$

Question: 6.

Compare answers (i) and (iv) above.

The answers are the same with the exception of the negative signs.

Question: 7.

Determine each of the following:

i. Taylor(e^{ix} , x, 8) [Make sure to use the complex number i.]

$$1+ix-\frac{x^2}{2!}-\frac{ix^3}{3!}+\frac{x^4}{4!}+\frac{ix^5}{5!}-\frac{x^6}{6!}-\frac{ix^7}{7!}+\frac{x^8}{8!}$$

ii. Real terms in: Taylor(e^{ix} , x, 8)

$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}$ Note the similarity with Taylor polynomial for cos(x)

iii. Imaginary terms in: Taylor(e^{ix} , x, 8)

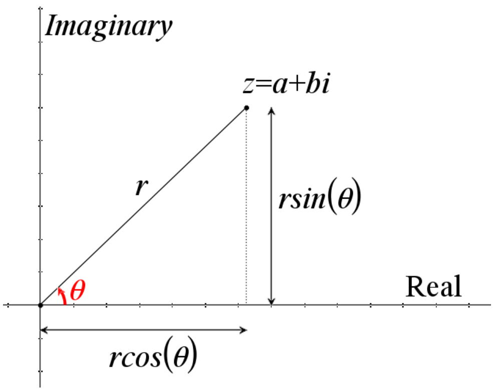
$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}$ Note the similarity with Taylor polynomial for sin(x)

Question: 8.

For each of the following $z=\sqrt{3}+i$

i. Calculate z^2 $2+2\sqrt{3}i$

ii. Calculate z^3 $8i$



Question: 9.

Use the diagram (right) to show that the complex number:

$z=a+bi$ (rectangular form) can be written in polar form:

$z=r\left(\cos(\theta)+i\sin(\theta)\right)$

Students should state that $a=r\cos(\theta)$ and $b=r\sin(\theta)$

therefore $z=a+bi=r\left(\cos(\theta)+i\sin(\theta)\right)$

Question: 10.

Explain the connection between: $z=re^{i\theta}$ and $z=rcis(\theta)$

Question 7 showed that the real part of $e^{i\theta}$ is equivalent to $\cos(\theta)$ and the imaginary part of $e^{i\theta}$ is the same as $\sin(\theta)$ therefore $e^{i\theta}=\cos(\theta)+i\sin(\theta)$ so it follows that $re^{i\theta}=r\left(\cos(\theta)+i\sin(\theta)\right)$

Question: 11.

Use the index laws to show that $z^n=r^n cis(n\theta)$

$\left(re^{i\theta}\right)^n=r^n e^{i.n\theta}$ since $re^{i\theta}=r\left(\cos(\theta)+i\sin(\theta)\right)$ then $\left(re^{i\theta}\right)^n=r^n\left(\cos(n\theta)+i\sin(n\theta)\right)$

Author: P. Fox

Question: 12.

Write $z=\sqrt{3}+i$ in polar form (either $z=re^{i\theta}$ or $z=rcis(\theta)$) and hence evaluate z^2, z^3 and z^6.

$z=\sqrt{3}+i=2e^{\frac{i.\pi}{6}}$ therefore $z^2=4e^{\frac{i.\pi}{3}}$ and $z^3=8e^{\frac{i.\pi}{2}}$ and $z^6=64e^{i\pi}=-64$

Author: P. Fox