Transformations 3

Reflections & snoitcelfeR

Teachers Teaching with Technology™
Professional Development from Texas Instruments

Student Activity

7 8 9 10 **11** **12**

TI-Nspire™ | Activity | Student | 1.5 hours

Introduction

The Transformations activities in this series involve using multiple functionalities of TI-Nspire to explore the concept of transformations of the plane – translations, dilations, reflections and combinations of these – and their effect on functions. The aim of this approach is to develop a sound understanding by linking graphical and algebraic representations of transformation problems and their solution and making sense of the algebra through visualisation of the relationship between the graph of the original function and the graph of the transformed function.

This is the third transformation activity in the series, and it focuses on reflections and on combinations of transformations.

Exploration 3a. Reflection in the y-axis

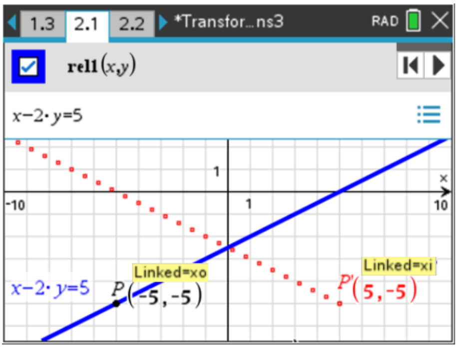
a. In this part of the activity, you will explore the image of the line $x - 2y = 5$ under the transformations:

• reflection in the y-axis

Please refer to the TI-Nspire document 'Transformation3' Open the TI-Nspire document 'Transformation3' and select page 2.1. On page 2.1, P is a point on the graph of the functional relationship with equation $x - 2y = 5$.

The equation of the graph has been entered as a relation, $\mathbf{rel1}(x, y)$ (i.e. 'Relation 1').

The coordinates of the points shown have been stored as the variables, $P(xo, yo)$ and $P'(xi, yi)$.

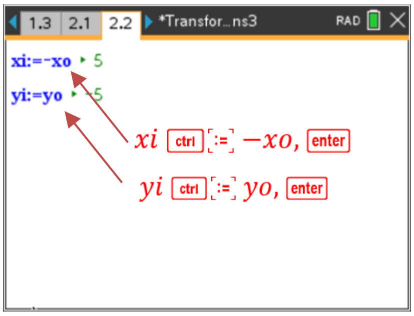


Technology Tip!

> An easy way to enter (or edit) the equation of a graph as a relation:
> [menu] > Graph Entry/Edit > Relation.

Making P' the image of P under the transformation

For P' to be the image of P, it needs to be reflected in the y-axis, therefore xi assigned the value $-xo$ and yi assigned the value yo (e.g. $P(-5, -5) \to P'(5, -5)$ because the sign of the x-coordinates are reversed, but the y-coordinates are unchanged by the reflection in the y-axis.



What to do – select page 2.2. This page is a Notes application with two Maths Boxes. Click ( ) the Maths Box so that a red dotted border appears, then input

* In the top box: $xi := -xo$ and press [enter].
* In the bottom box: $yi := yo$ and press [enter].

Author: Frank Moya

The coordinates of P' are now $(-xo, yo)$. When point P is moved along the graph on page 2.1, P' will remain the image of P under the transformation.

Graph and equation of the transformed function

P can be animated using the control buttons. Alternatively, P can be grabbed (ctrl + click) and dragged along the graph. Press esc to release the 'grabbed' P.

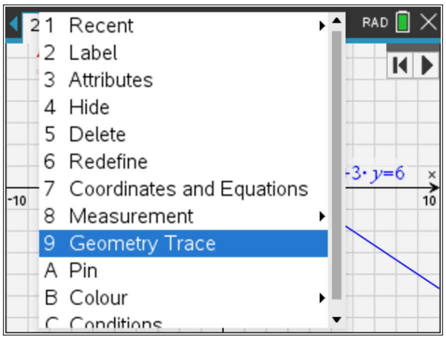
Question 1

On page 2.1, use the control buttons to move point P, and observe the motion of point P'.
Describe the relationship between the positions of P and P' as the points move.

Sample response. P' is the mirror-image (in the y-axis) of P, and it traces a linear path with a negative gradient that has the same 'steepness' as the gradient as the graph of $x - 2y = 5$.

Path of P'

Return P to the original position by clicking the left control button. The 'Geometry Trace' tool will be used to obtain an outline of the path of P'. What to do. To activate this tool, move the cursor to the point P' and open the context menu by pressing ctrl + menu. Select 'Geometry Trace', then animate P. A trace of P' will be visible. When P' is outside the window settings, return P' to the starting position, then exit the tool by pressing esc.



Question 2

Describe qualitatively some of the key features of the trace of P'. How do these key features relate to the graph of the original function?

Sample response. P' traces a line that mirrors the graph of $x - 2y = 5$, intersecting this graph and the y-axis at the point with coordinates $\left(0, -\frac{5}{2}\right)$. However, whereas $x - 2y = 5$ has a positive gradient and x-intercept at $(5,0)$, the trace of P' has a negative gradient and x-intercept at $(-5,0)$.

Question 3

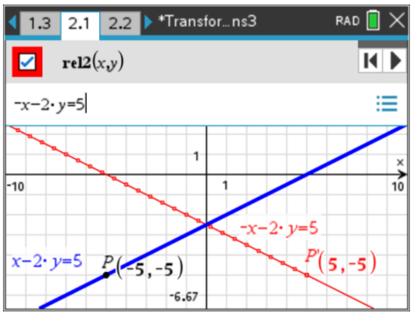
a. Use the trace of P' to determine the equation of the transformed function.

b. Test whether your equation in part b. is correct by graphing your equation on page 2.1, using the same set of axes as the graph of the original function. You should enter your equation as 'Relation 2' (menu > Graph Entry/Edit > Relation will ensure 'Relation' is selected. (The graph of your equation should contain all points traced by P').

c. Find (i) the gradient and (ii) the coordinates of the axes intercepts of the graph of the transformed function.

a. $-x - 2y = 5$ or $y = -\frac{x}{2} - \frac{5}{2}$ (or equivalent)

b. See screenshot on the right



c. i. gradient $= \frac{-1}{2}$ (by using $\frac{rise}{run}$ on graph, or rearranging eqn.)

ii. When $y = 0, x = -5$. When $x = 0, y = -\frac{5}{2}$.

x-intercept $(-5,0)$ y-intercept $\left(0, -\frac{5}{2}\right)$

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Relationship between equation of the transformed and original functions

Now consider how the equation of the transformed function could have been obtained from the equation of the original function, $x - 2y = 5$.

Question 4

Which one of the following is the correct equation for the transformed function? Test your answer by entering the equation as 'Relation 2' on page 2.1. The graph should contain all points on the trace of P'.

A. $\boldsymbol{-x - 2y = 5}$ (correct)
B. $-x + 2y = 5$
C. $x + 2y = 5$
D. $x - 2y = 5$

Explaining the equation of the transformed function

We saw that the point $P(x, y)$ maps to point $P'(-x, y)$ under this transformation. That is, if P' has coordinates (x', y'), then

$x' = -x$ (*eqn1*)
$y' = y$ (*eqn2*)

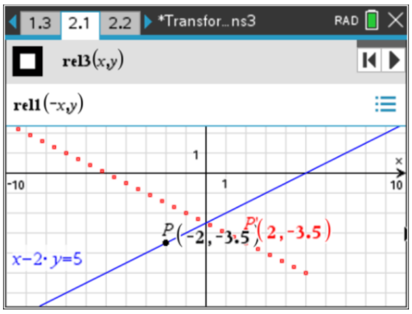
Question 5

Use the equations in blue above to explain why the graph of the transformed function - which contains all points on the trace of P' - can be expressed as $-x - 2y = 5$.

Rearrange *eqn1* to $x = -x'$ and *eqn2* to $y = y'$. Substitute in $x - 2y = 5$.
$-x' - 2y' = 5$. Hence transformed function is $-x - 2y = 5$.

Question 6

On page 2.1, navigate to the Graph Entry (menu > Graph Entry/Edit > Relation). Turn off 'Relation 2' and for 'Relation 3' input $\mathrm{rel1}(-x, y)$, as shown on the right.



a. What is the relationship between the graph of 'Relation 3' and the trace of P' ? 'Relation 3' should contain all points on the trace of P'.

b. Confirm that the equation of 'Relation 3' is equivalent to the equation in Question 6 above.

$\mathrm{rel1}(x, y)$ is $x - 2y = 5$ therefore
$\mathrm{rel1}(-x, y)$ is $-x - 2y = 5$

Exploration 3b. Reflection in the y-axis followed by reflection in x-axis

Navigate to page 3.1 of the TI-Nspire file, Transformations3. The graph of $x - 2y = 5$ is shown in blue. The reflection of this graph in the y-axis has the equation $-x - 2y = 5$, and is shown in red. On page 3.1, P is a point on the graph of $-x - 2y = 5$. The equation of the graph has been entered as a relation, $\mathbf{rel2}(x, y)$ (i.e. 'Relation 2'). The coordinates of the points shown have been stored as the variables, $P(xo, yo)$ and $P'(xi, yi)$.

You will now explore the image of the line $x - 2y = 5$ under the transformations:
• reflection in the y-axis
(This transformed function has equation $-x - 2y = 5$)
Followed by
• reflection in the x-axis (i.e. the graph of $-x - 2y = 5$ is reflected in the x-axis).

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What to do – select page 3.2. This page is a Notes application with two Maths Boxes. Click () the Maths Box so that a red dotted border appears, then input
* In the top box: $xi := xo$ and press [enter].
* In the bottom box: $yi := -yo$ and press [enter].

That is, point $P(x, y)$ maps to point $P'(x, -y)$ under this transformation. If P' has coordinates (x', y'), then

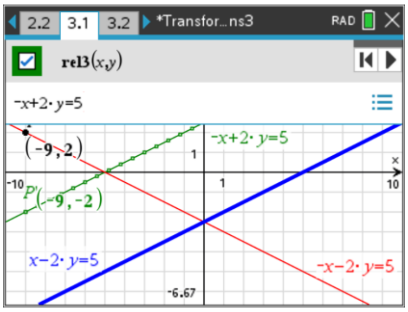
$$x' = x$$
$$y' = -y$$

Trace of P'

The 'Geometry Trace' tool will again be used to obtain an outline of the path of P'.
What to do. Activate this tool by moving the cursor to the point P' and open the context menu by pressing [ctrl]+[menu]. Select 'Geometry Trace', then animate P. When P' is outside the window settings, return P' to the starting position, then exit the tool by pressing [esc].

Question 7

a. Use the trace of P' to determine the equation of the transformed function.
b. Test whether your equation in part b. is correct by graphing your equation on page 3.1, using the same set of axes as the graph of the original function. You should enter your equation as 'Relation 3' ([menu] > Graph Entry/Edit > Relation will ensure 'Relation' is selected. (The graph of your equation should contain all points traced by P').
c. Find (i) the gradient and (ii) the coordinates of the axes intercepts of the graph of the transformed function.



a. $-x + 2y = 5$ or $y = \frac{x}{2} + \frac{5}{2}$ (or equivalent)

b. See screenshot on the right

c. i. gradient $= \frac{1}{2}$ (by using $\frac{rise}{run}$ on graph, or rearranging eqn.)

ii. When $y = 0, x = -5$. When $x = 0, y = \frac{5}{2}$.

x-intercept $(-5,0)$ y-intercept $\left(0, \frac{5}{2}\right)$

Question 8

On page 3.1, navigate to the Graph Entry ([menu] > Graph Entry/Edit > Relation). Turn off 'Relation 3' and for 'Relation 4' input $\text{rel1}(-x, -y)$, as shown on the right. Write the equation of $\text{rel1}(-x, -y)$ and explain why this contains all points on the trace of P'.

$\text{rel1}(x, y)$ is $x - 2y = 5$ therefore
$\text{rel1}(-x, -y)$ is $-x - 2(-y) = 5$, therefore $-x + 2y = 5$ (which is a reflection in the y-axis and a reflection in the x-axis).

Generalising the equation of transformed functions involving reflections

You will now consider the form of the equation of the transformed function for any functional relation, whether linear or non-linear.
You will explore the equations of the transformed functions for all the cases in **Table 1** below.

What to do. Step 1. Erase old Geometry Trace. Navigate to page 4.1. Erase the old geometry trace before starting the next problem from Table 1.

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Technology Tip!

Erasing the 'Geometry Trace': move the cursor to a blank part of the graphing window and open the context menu (ctrl + menu). Select 'Erase Geometry Trace'.

Step 2. Edit Equation. On page 4.1, replace the equation of 'Relation 1' with the equation of the original function for the next problem in Table 1.
For example, second problem from Table 1: Edit 'Relation 1' to $y = \sqrt{4 - x} + 1$, as shown on the right.

Step 3. Edit reflections. On page 4.2, edit the Maths Boxes to match the set of transformations listed for that problem in Table 1.

Step 4. Obtain a trace of P' .

Return P to the original position by clicking the left control button.
Activate the Geometry Trace tool: move the cursor to the point P' and open the context menu by pressing ctrl + menu. Select 'Geometry Trace', then animate P by clicking the right control button. A trace of P' will be visible. When P' is outside the window settings, return P' to the starting position, then exit the tool by pressing esc.

Technology Tip!

Selecting the correct point when P and P' are close together. Move the cursor close to the point you wish to select. Press the tab key until the desired point is displayed. (Repeated use of the <tab> key will cycle through all nearby points and other object).

Step 5. Equation of transformed function: its graph should contain all points on the trace of P'.

- Write down an equation which you believe corresponds with the transformed function.
- Test whether your equation contain all points on the trace of P'. On page 4.1, enter your equation for the transformed function as 'Relation 2'.
- On page 4.1, also enter the transformed function as 'Relation 3' in the form: $\text{rel1}(a \cdot x, b \cdot x)$, where $a = \pm 1$ and $b = \pm 1$. The graph of 'Relation 3' should also contain all points on the trace of P'.

Step 6. Use algebra to confirm the equation of the transformed function.
For reflections in axes, the point $P(x, y)$ maps to point $P'(\pm 1 \cdot x, \pm 1 \cdot y)$; that is, if P' has coordinates (x', y'), then $x' = \pm 1 \cdot x$ and $y' = \pm 1 \cdot y$.
Confirm algebraically the equation of the transformed function.
Repeat Steps 1 – 6 for the next problem in Table 1.

Question 9
Follow Steps 1 – 6 above for each original equation below and complete the table.

Original equation	**Dilation of factor**: TABLE 1	Equation of transformed function
a. $2x - 5y - 10 = 0$	a. Reflection in x-axis followed by reflection in y-axis b.	a. $2x - 5(-y) - 10 = 0$ b. $-2x + 5y - 10 = 0$
b. $y = \sqrt{4 - x} + 1$	Reflection in y-axis	$y = \sqrt{4 - (-x)} + 1$ $y = \sqrt{4 + x} + 1$

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c. $y = \sqrt{4-x} + 1$	Reflection in x-axis	$(-y) = \sqrt{4-x} + 1$ $y = -\sqrt{4-x} - 1$
d. $y = \sqrt{4-x} + 1$	a. Reflection in y-axis followed by b. reflection in x-axis	a. $y = \sqrt{4-(-x)} + 1$ b. $(-y) = \sqrt{4-(-x)} + 1$ $y = -\sqrt{4+x} - 1$
e. $y = \frac{(x+4)^2-2}{2}$	Reflection in x-axis	$(-y) = \frac{(x+4)^2-2}{2}$ $y = \frac{-(x+4)^2+2}{2}$
f. $y = \frac{(x+4)^2-2}{2}$	Reflection in y-axis	$y = \frac{((-x)+4)^2-2}{2}$ $y = \frac{(4-x)^2-2}{2}$
g. $y = \frac{(x+4)^2-2}{2}$	a. Reflection in x-axis followed by b._reflection in y-axis	a. $y = \frac{-(x+4)^2+2}{2}$ b. $y = \frac{-((-x)+4)^2+2}{2}$ $y = \frac{-(4-x)^2+2}{2}$

Question 10
Show all working used in Step 6 above to obtain the equation of the transformed function using algebra. Graph your equation to verify that it is an equivalent form of the equation from Step 5.
Substitutions shown in column 3 of Table 1.

Exploration 3c. Combinations of transformations

In the transformations considered so far, the **order** in which the sequence of transformations was carried out made no difference to the final equation of the transformed function. You will now consider some cases where order does matter.

3c.i. Reflection in y-axis followed by a translation in the direction of the x-axis

The function with equation $4x + 5y = 10$ undergoes the following sequence of transformations:

- reflection in the y-axis
  followed by
- translation of 3 units in the direction of the positive x-axis (i.e 3 units right).

You will explore this sequence of transformations graphically, using the instructions below.

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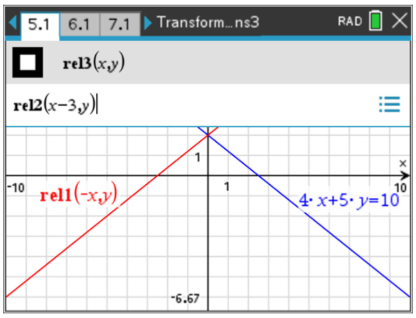
What to do. Navigate to page 5.1 of the TI-Nspire document. The graph of $4x + 5y = 10$ has been entered as 'Relation 1' ($\mathbf{rel1}(x, y)$). *
Consider '**reflection in the y-axis**'. Navigate to the Graph Entry (menu > Graph Entry/Edit > Relation). For 'Relation 2' ($\mathbf{rel2}(x, y)$) input $\text{rel1}(-x, y)$. * Now consider '**followed by translation of 3 units in the direction of the positive x-axis**'.



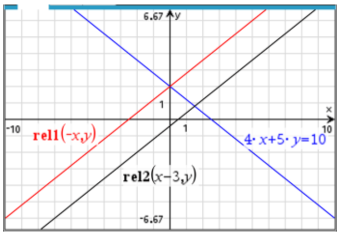
Navigate to the Graph Entry (menu > Graph Entry/Edit > Relation). For 'Relation 3' ($\mathbf{rel3}(x, y)$) input $\mathbf{rel2}(x - 3, y)$.

Question 11

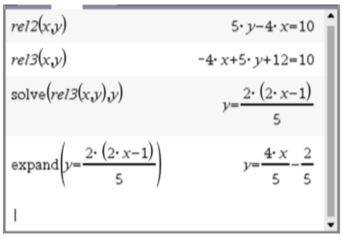
a. Determine the equation of 'Relation 2' (the equation after the reflection) and the equation of 'Relation 3' (the equation after the reflection followed by the translation).
Express the equation to 'Relation 3' in the form $y = m{\cdot}x + c$.
'Relation 2' is $4(-x) + 5y = 10$ or $-4x + 5y = 10$
'Relation 3' is $4\big(-(x-3)\big) + 5y = 10$ or $-4(x-3) + 5y = 10$ or $y = \frac{4}{5}x - \frac{2}{5}$

Check your equations as follows. Add a Calculator page as page 5.2 (press ctrl + page > Add Calculator). In the calculator page, input $\mathbf{rel2}(x, y)$ and <enter>. Then input input $\mathbf{rel3}(x, y)$ and <enter>. Solve the equation for y in order to express the answer in the form $y = m{\cdot}x + c$.

b. Explain why $\mathbf{rel3}(x, y)$ gives the final equation of the transformed function.
See substitutions in answers to Question 11a. above



Page 5.1



Page 5.2

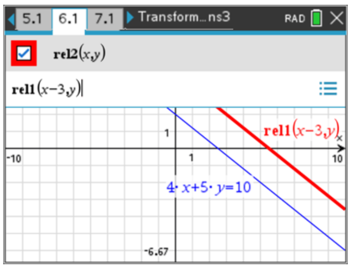
3c.ii. Translation in the direction of the x-axis followed by reflection in y-axis

You will now explore whether reversing the order in which this sequence of transformation is applied produces the same or a different transformed function.

For this exploration, the function with equation $4x + 5y = 10$ undergoes the following sequence of transformations:

- translation of 3 units in the direction of the positive x-axis (i.e 3 units right)
  followed by
- reflection in the y-axis.

What to do. Navigate to page 6.1 of the TI-Nspire document. The graph of $4x + 5y = 10$ has been entered as $\mathbf{rel1}(x, y)$. Consider '**translation of 3 units in the direction of the positive x-axis**'.
Navigate to the Graph Entry (menu > Graph Entry/Edit > Relation).
For 'Relation 2' ($\mathbf{rel2}(x, y)$) input $\text{rel1}(x - 3, y)$.



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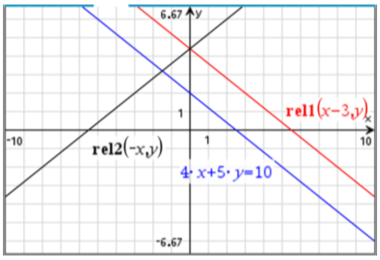
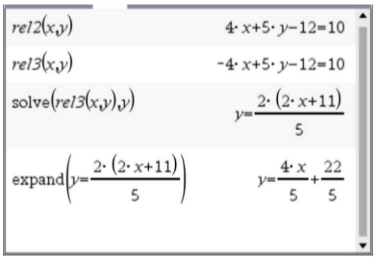
* Now consider '**translation 3 units in the direction of the positive x-axis followed by reflection in the y-axis.**
Navigate to the Graph Entry (menu > Graph Entry/Edit > Relation). For 'Relation 3' ($\mathbf{rel3}(x,y)$) input $\mathbf{rel2}(-x,y)$.

Question 12

a. Determine the equation of 'Relation 2' (the equation after the translation) and the equation of 'Relation 3' (the equation after the translation followed by the reflection).
Express the equation to 'Relation 3' in the form $y = m{\cdot}x + c$.
'Relation 2' is $4(x-3)+5y=10$ or $4x+5y=22$
'Relation 3' is $4(-x)+5y=22$ or $y=\frac{4}{5}x+\frac{22}{5}$

Check your equations as follows. Add a Calculator page as page 6.2 (press ctrl + +page > Add Calculator). In the calculator page, input $\mathbf{rel2}(x,y)$ and <enter>. Then input input $\mathbf{rel3}(x,y)$ and <enter>. Solve the equation for y to express the answer in the form $y = m{\cdot}x + c$.





b. Explain why (for this sequence of transformations) reversing the order in which the transformations are applied results in a different outcome.

Sample response. The order in which the transformations were applied changed the outcome because one transformation produces an image upon which the other transformation is then performed.
In this case, if the translation is applied first, then the x-intercept will be at $\frac{11}{2}$. On reflection in the y-axis, the x-intercept will be at $-\frac{11}{2}$.
However, if the reflection is applied first, then the x-intercept will be at $-\frac{5}{2}$. Then, following the translation 3 units right, the x-intercept will be at $\frac{1}{2}$. Hence the transformed functions are different depending on the order in which this sequence of transformations is applied.

Question 13

Consider the function with equation $y=\sqrt{6-x}$ undergoing the following sequence of transformations:

- translation of 2 units in the direction of the negative x-axis (i.e 2 units left)
  followed by
- dilation of factor $\frac{1}{2}$ from the y-axis (i.e. parallel to the x-axis).

a. Determine the equation of the transformed function.

Original function	Apply translation	Followed by dilation (Transformed function)
$y=\sqrt{6-x}$	$y=\sqrt{6-(x+2)}$	$y=\sqrt{4-\left(\frac{x}{\left(\frac{1}{2}\right)}\right)}$ or $y=\sqrt{4-2x}$

b. Navigate to page 7.1 of the TI-Nspire document. Use a method similar to that used in Question 11 above to check your equation.

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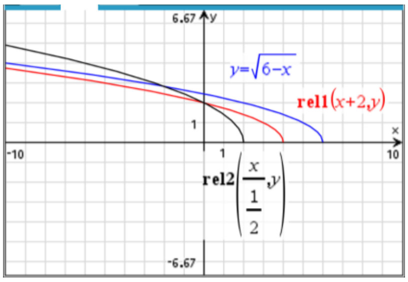
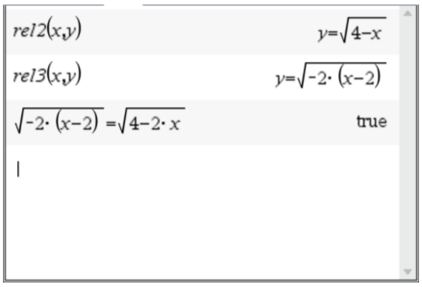




Question 14

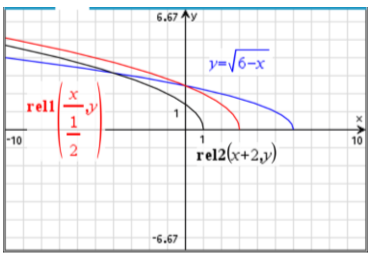
Now consider the function with equation $y = \sqrt{6-x}$ undergoing the sequence of transformations in reverse order. That is:

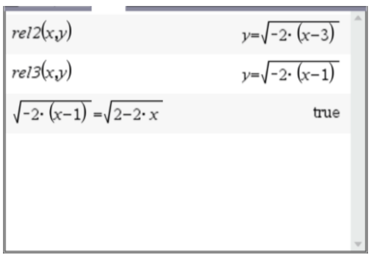
- dilation of factor $\frac{1}{2}$ from the y-axis (i.e. parallel to the x-axis)
  followed by
- translation of 2 units in the direction of the negative x-axis (i.e 2 units left).

a. Determine the equation of the transformed function.

Original function	Apply dilation	Followed by translation (Transformed function)
$y = \sqrt{6-x}$	$y = \sqrt{6 - \left(\frac{x}{\left(\frac{1}{2}\right)}\right)}$	$y = \sqrt{6 - \left(\frac{x+2}{\left(\frac{1}{2}\right)}\right)}$ or $y = \sqrt{6-2(x+2)}$ or $y = \sqrt{2-2x}$

b. Navigate to page 8.1 of the TI-Nspire document. Use a method similar to that used in Question 12 above to check your equation.





c. Explain why (for this sequence of transformations) reversing the order in which the transformations are applied results in a different transformed function.

Sample response. The order in which the transformations were applied changed the outcome because one transformation produces an image upon which the other transformation is then performed.

In this case, if the translation is applied first, then the x-intercept will be at 4. Then dilation of factor $\frac{1}{2}$ the y-axis 'squashes' the graph, so that the x-intercept will then be at 2 .

However, if the dilation is applied first, then the 'squashing' of the graph results in the x-intercept at 3. Then, following the translation 2 units left, the x-intercept will be at 1 . Hence the transformed functions are different depending on the order in which this sequence of transformations is applied.

Authors: Frank Moya