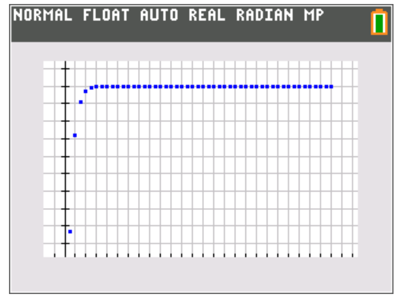
Summing Up Geometric Series

Math Objectives

- Students will understand how to graph an infinite geometric series and how to analyze the graph.
- Students will understand and justify the sum of an infinite geometric series.
- Students will be able to explain why the sum of an infinite geometric series is a finite number if and only if $|r| < 1$.
- Students will try to make a connection with how to understand these topics in IB Mathematics courses and on their final assessments.



Vocabulary

- Geometric Series
- Infinite series
- Ratio of a geometric series
- Sigma notation

About the Lesson

- This lesson is aligning with the curriculum of IB Mathematics Applications and Interpretations SL/HL and IB Mathematics Approaches and Analysis SL/HL
- This falls under the IB Mathematics Core Content Topic 1 Numbers and Algebra:

  1.3: **(a)** Geometric sequences and series

  (b) Use of the formulae for the n^{th} term and the sum of the 1^{st} n terms of the sequence

  (c) Use of sigma notation for the sums of geometric sequences

  AI HL 1.11: The sum of infinite geometric sequences

  AA 1.9: Sum of infinite convergent geometric sequences

As a result, students will:

- Apply this information to real world situations.

Teacher Preparation and Notes.

- This activity is done with the use of the TI-84 family as an aid to the problems.

Activity Materials

- Compatible TI Technologies: TI-84 Plus*, TI-84 Plus Silver Edition*, TI-84 Plus C Silver Edition, TI-84 Plus CE

Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family. Slight variations to these directions may be required if using other calculator models.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:

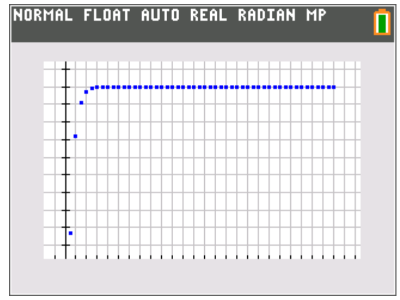
Student Activity

84CE-SummingUpGeometricSeries-Student.pdf

84CE-SummingUpGeometricSeries-Student.doc

** with the latest operating system (2.55MP) featuring MathPrint™ functionality.*

In this activity, you will explore the concept of finding the sum of an infinite geometric series. Reviewing the concepts of when you can find the sum of an infinite geometric series will be the first task, discussing with your classmates not only what the characteristics of a geometric sequence are, but also the key characteristic that allows you to add every term of the infinite sequence and still get a non-infinite sum.



Let us review the characteristics of a geometric sequence. A **geometric sequence** is a sequence of terms where the ratio of every two consecutive terms is constant. The constant ratio is referred to as the **common ratio** or ***r***. To find subsequent terms of a geometric sequence, multiply a term by ***r***. The nth term, u_n, formula for a geometric sequence is $\boldsymbol{u_n = u_1 \cdot r^{n-1}}$, where u_1 is the first term, and *r* is the common ratio.

> **Teacher Tip:** Students need to also be aware of the finite geometric series formula $\left(S_n = \frac{u_1(1-r^n)}{1-r}\right)$ and the infinite geometric series formula $\left(S_\infty = \frac{u_1}{1-r}\right)$ that will be discussed in questions throughout the activity.

Problem 1 – Geometric Sequence Practice

1. Find the next three terms of each infinite geometric series.

   (a) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$
   Solution: $\frac{1}{16}, \frac{1}{32}, \frac{1}{64}$

   (b) $2+\frac{3}{2}+\frac{9}{8}+\cdots$
   Solution: $\frac{27}{32}, \frac{81}{128}, \frac{243}{512}$

2. Discuss with a classmate how you would find the next three terms of each series. Explain your results.

   Solution: First you would find the common ratio by dividing each term by its previous term, then you would multiply this common ratio be the third, fourth and fifth terms.

3. Sigma notation is used at times to express a series. The symbol for sigma, Σ, actually means the sum of. Using the nth term formula from above and the sigma notation, write an expression in terms of *n* that describes each of the series from number 1.

Solution: (a)

$$\sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{2}^{n-1}$$

(b)

$$\sum_{n=1}^{\infty} 2 \cdot \frac{3}{4}^{n-1}$$

4. Discuss with a classmate how you would find the sum of each series in number 1. Explain your results.

   Possible Discussions: Depending on if you were finding a partial sum or an infinite sum, you could use either of the geometric series formulas. If it is a partial sum, you could simply add the terms. You could use your handheld and sigma notation if you are finding a partial sum.

Problem 2 – Finding the Sum of a Geometric Series

There are two types of geometric series. There is the partial sum or finite series and then there is the infinite series.

1. Discuss with a classmate the formula to find a partial sum of a geometric series. Explain what you would need to use the formula and if there are any restrictions.

   Solution: Formula: $S_n = \frac{u_1(1-r^n)}{1-r}$

   You will need the first term, the common ratio and how many first terms will be added. There could be restrictions if r > 1 and you are finding the sum of all the terms.

2. Explain how you could use sigma notation to find the partial sum of a geometric series as well. Explain what you would need to use sigma.

   Solution: You would need to know how many terms you are adding together or which term you are starting with and ending with (x), you would need the common ratio and the first term. This information would be used to substitute in:

$$\sum_{n=1}^{x} u_1 \cdot r^{n-1}$$

3. Discuss with a classmate the formula to find an infinite sum of a geometric series. Explain what you would need to use the formula and if there are any restrictions.

Solution: Formula: $S_n = \frac{u_1}{1-r}$

You will need the first term and the common ratio.

The restriction is if r > 1, then you cannot find the infinite sum, as it would be infinite.

4. Explain how you could use sigma notation to find the infinite sum of a geometric series as well. Explain what you would need to use sigma.

Solution: You would need to know the first term and the common ratio to substitute in:

$$\sum_{n=1}^{\infty} u_1 \cdot r^{n-1}$$

5. Given the geometric sequence 1, -4, 16, -64, …, find the partial sum of the first 9 terms.

Solution: $S_9 = 589{,}824$

6. Given the geometric sequence 9, 3, 1, $\frac{1}{3}$, …, find the infinite sum.

Solution: $S_\infty = 13.5$

7. Write the sequences in questions 5 and 6 in sigma notation. Explain if you can use your handheld to verify your answers using sigma notation.

Solution: (5)

$$\sum_{n=1}^{9} 1 \cdot (-4)^{n-1} = 589{,}824$$

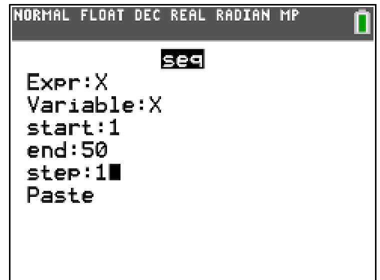
(6)

$$\sum_{n=1}^{\infty} 9 \cdot \left(\frac{1}{3}\right)^{n-1} = 13.5$$

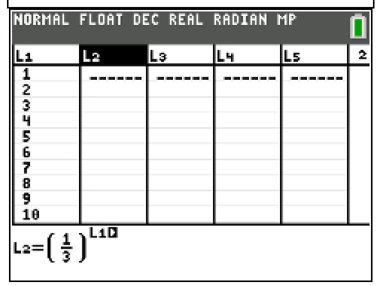
Number 5 can be done on the 84, but number 6 cannot.

Problem 3 – Visualizing an Infinite Geometric Series

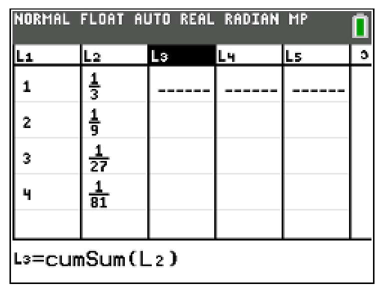
Use Lists to display the terms of each series.



Press **stat**, **enter** to access the table of data screen. In **L1**, enter **seq(x,x,1,50)** in the top most cell. The **seq(** command can be found by pressing **2nd**, **stat [list]** and arrowing over to **OPS** and selecting **5:seq(**. Enter the information in the **seq** exactly as shown in the screen to the right.



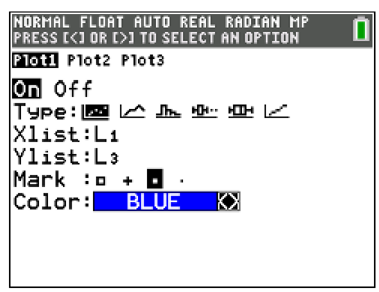
In the top most cell of **L2**, type $\left(\frac{1}{3}\right)^{L_1}$ and **enter**.



Next, we will graph the series.

First, we will need to generate a list with the cumulative sums of the terms of the sequence. To do this, move to the top most cell of **L3**, press **enter**, then press **2nd**, **stat [list]** and arrow over to **OPS** and select **6:cumSum(**. Then type **2nd**, **2 [L2]** and press **enter**.

This will list the first 50 partial sums of the series in **L3**.



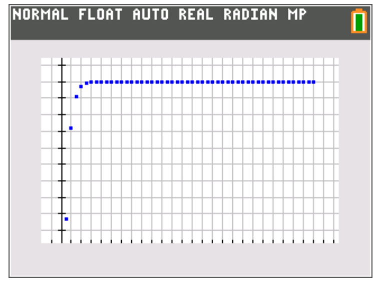
You can view a graph for each series by creating a scatter plot of the values of the partial sums of the series.

To create a scatter plot, select **2nd**, **y= [stat plot]**, **1**.

Set up as shown in the figure to the right.

To view the graph, press **zoom**, **9:ZoomStat**.

To get an even better view of the behavior of the partial sums, you can change the scaling of the *x* and *y*-axes. Press window and change each of the following: **Xscl:** 2 **Yscl:** 0.2**.** The graph should look like the screen shown to the right.



1. Discuss with a classmate the characteristics you notice about the graph of the partial sums. Write down what you notice.

   Possible Discussions: The graph of the partial sums seems to be plateauing, almost as if it was approaching an asymptote. The partial sums are getting smaller and smaller as if they are not affecting the sum. This plateau or asymptote could be the infinite sum of all the terms.

2. Discuss with a classmate what you expect the total sum of all of the terms of the geometric series would be. Explain how you arrived at your conclusion.

   Solution: The terms of the series follow the pattern $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$, so the sum will be greater then one third, but since the successive terms are getting smaller and smaller, the total sum will only be slightly higher than one third, roughly one half.

   Teacher Tip: Have students convert these fractions into decimals and then have them add them together. By expressing their answers in decimal form, students can see how close the sum is getting to $\frac{1}{2}$, even after only a few terms are used.

3. Write an expression for the sum of the infinite series. Find the value of this sum. Explain how you found this sum.

   Solution: The sum of the infinite series would be:
   $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$
   Thus, $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots=\frac{1}{2}$

4. Express your answer from Question 6 in sigma notation.

   Solution: Expressed in sigma notation, we have

$$\sum_{n=1}^{\infty}\frac{1}{3}\cdot\left(\frac{1}{3}\right)^{n-1}=\frac{1}{2}$$

5. Suppose you change the base of $\left(\frac{1}{3}\right)$ to a 3.

a. Express the sum of the terms as an infinite sum.

Solution: If we doubled the side of the square, we would have 2 + 4 + 8 + 16 +…

b. Describe what happens to this sum as each term increases. Explain your answer.

Solution: If we continued to increase a side of the square, the sum would get infinitely large.

6. Give an example of an infinite geometric series that you think would have a finite sum and an example of one that you think would not have a finite sum. Explain your reasoning.

Solution: For an infinite geometric series to have a finite sum, the common ratio (*r*) must be a proper fraction, i.e., $|r| < 1$. Possible examples include 25 + 5 + 1 + 0.2 + … or
0.1 + 0.01 + 0.001 + 0.0001 + …
If $|r| \geq 1$, there would not be a finite sum. Possible examples include 1 + 4 + 16 + 64 + …

7. Based on the information above, describe what conjecture that must be true about the ratio of the consecutive terms of an infinite geometric series for the series to have a finite sum.

Solution: The ratio, *r*, is conjectured to be a proper fraction, i.e., $|r| < 1$.

8. Find the values of the ratio *r* where an infinite geometric series appears to have a finite sum.

Solution: The ratio *r* would appear to be a proper fraction, i.e., $|r| < 1$.

Further IB Application

A local coffee shop had an amazing first year after it opened, earning $40,000 of profit. Unfortunately, the profits have been decreasing by 10% each year after the first. Assuming that this trend continues, find the total profits the shop hopes to earn over the course of its lifetime.

Solution: $S_\infty = \frac{40000}{1-0.90} = \$400{,}000$ or

$$\sum_{n=1}^{\infty} 40000 \cdot (0.90)^{n-1} = \$400{,}000$$

Teacher Tip: Throughout this activity, the students are asked to discuss with classmates and explain how they achieved their answers. This is a wonderful opportunity to create a student led classroom. As you float around the room, listen to what they are saying, add to their discussions, and give them leading questions to see how they respond.

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