Complex Spirals

Teacher Notes & Answers

7 8 9 10 **11** 12

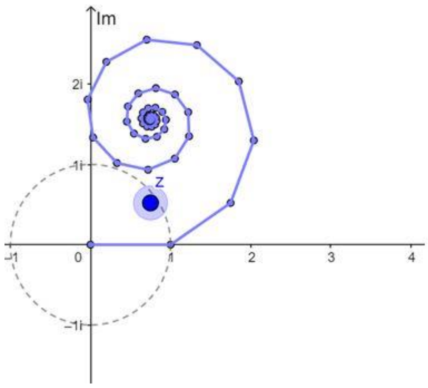
TI-Nspire™ Activity Student 4 hours

Instructioduction

Complex numbers can be expressed in different forms. We can also create a geometric series consisting of the powers of complex numbers. When we graph the power series of complex numbers on the Argand diagram, we can observe various patterns. You will use your TI Nspire CX II CAS to investigate those patterns and establish the conditions under which they occur.

Teacher Notes:

This document outlines reasoning behind the activity associated with the Complex Spirals Guided Exploration.



This is a guided exploration for Specialist Mathematics Unit 1. This task begins in a scaffolded manner and then becomes more open-ended, with a focus on students generalising their results and/or observations. Students are encouraged to comment and investigate patterns that they see arise from their work. Students are to select different complex numbers throughout the task, and as such the teacher should anticipate that students will obtain different results, however the teacher may see that students record similar observations.

For teachers there is a 'complete' version of the Complex Spirals TI-Nspire file. This file generates the geometric sequence, series, and limiting value from given (initial) complex number. This makes it quicker to mark student work!

A feature of this task is that students will learn how to use their TI-Nspire to perform operations with Complex Numbers. Teachers are encouraged to adjust the task to suit their students. For some teachers, questions 6 & 7 may be extension, while others may wish to include only question 6 and/or question 7.

The task could be implemented within the classroom, with teacher guidance, or given to students to complete outside of the classroom, or a combination of both.

Assumed knowledge:

- Geometric Sequences
- Infinite Geometric Series
- Complex Numbers (Cartesian form)
- Argand Plane

Note: The task does not assume knowledge of the polar form. In the Student Video, we introduce the concept of modulus as the distance from the origin and the polar angle, in both cases by using the CAS calculator. By looking at the powers of complex numbers and investigating the moduli and angles, students are led towards De Moivre's Theorem, if it has not been introduced previously.

Authors: James Mott & Bozenna Graham

Instructions – Complex powers and complex series.

Technology Tips!

To define a complex number in a Calculator Application, select [:=].
Use the template: | □ | for modulus or use the menus.
Menu > Algebra Number > Complex Number Tools > Magnitude.
Double tap the π key to access the complex number i.

Question: 1.

For each of the following: $u = 1 + i$.

a) Find modulus $|u|$.

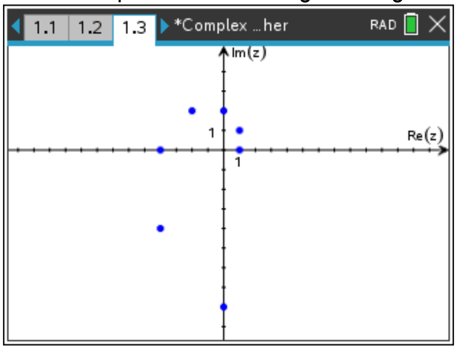
Answer: $|u| = \sqrt{2}$

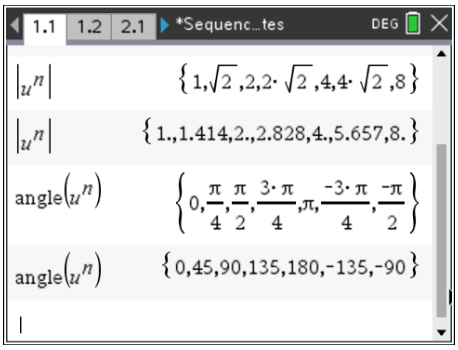
b) Use your TI-Nspire to find $u^0, u^1, u^2, u^3, u^5, u^6$ in Cartesian form.

Answers:

$u^0 = 1$
$u^1 = 1 + i$
$u^2 = 2i$
$u^3 = -2 + 2i$
$u^4 = -4$
$u^5 = -4 - 4i$
$u^6 = 8i$

c) Plot those powers on the Argand diagram.





d) What do you notice?

Answer: Student observations may vary. Possible responses may include:

- The points form a spiral pattern (anti clockwise)
- The points are spiralling outwards
- The moduli of the powers are increasing by the same factor
- The points are separated by the same angle (or increase by the same angle)

e) What do you think would happen if the pattern were to continue?

Answer: Student observations may vary here. Some possible responses may include:

- The spiral pattern will continue to grow and become larger
- The modulus of the powers will become larger, and grow to be of 'infinite length'
- The points are separated by the same angle (or increase by the same angle)

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f) The numbers $u^0, u^1, u^2, u^3, u^5, u^6, \ldots$ follow a geometric sequence.

i) State the first term of the sequence.

Answer: u^0 or 1

ii) ii) State the common ratio of the sequence.

Answer: u or $1+i$

Question: 2.

For each of the following: $w = \frac{1}{2} + \frac{1}{2}i.$

a) Find $|w|$.

Answer: $\frac{\sqrt{2}}{2}$

b) Use your TI Nspire to find $w^0, w^1, w^2, w^3, w^5, w^6$ in Cartesian form.

Answers:

$w^0 = 1$

$w^1 = \frac{1}{2} + \frac{i}{2}$

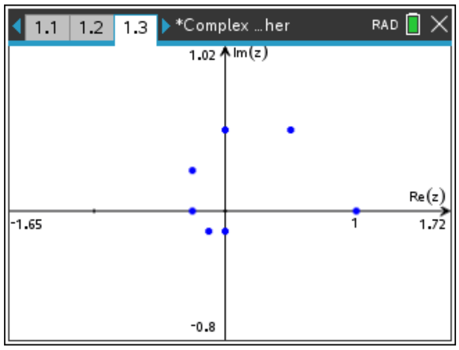
$w^2 = \frac{i}{2}$

$w^3 = -\frac{1}{4} + \frac{i}{4}$

$w^4 = -\frac{1}{4}$

$w^5 = -\frac{1}{8} - \frac{i}{8}$

$w^6 = -\frac{i}{8}$

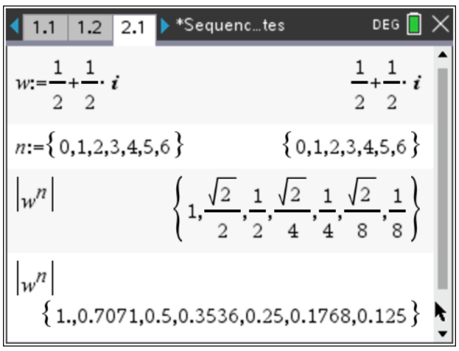


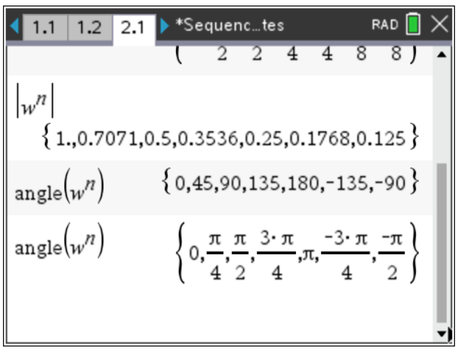
c) Plot those powers on the Argand diagram.

d) What do you notice?

Answer: Student observations may vary here. Some possible responses may include:

- The points form a spiral pattern (anti clockwise)
- The points are spiralling inwards
- The moduli of the powers are decreasing by the same factor
- The points are separated by the same angle (or increase by the same angle)





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e) What do you think would happen if the pattern were to continue?

Answer: Student observations may vary here. Some possible responses may include:

- The spiral pattern will continue to spiral towards zero
- The modulus of the powers will become smaller, and eventually becomes 'zero length'
- The points are separated by the same angle (or increase by the same angle)

f) Is there anything similar and/or different between the results from question 1 and question 2?

Answer: Student observations may vary here. Some possible responses may include:
Similarities:

- Both have spiral patterns and are spiralling anticlockwise
- The points are separated by the same angle (or increase by the same angle)
- Both spirals start at the point 1 on the real axis

Differences:

- One spiral is increasing/growing in size while the other spiral is decreasing/decaying in size
- The moduli of the powers are behaving differently. In one spiral they are getting bigger, yet in the other spiral they are getting smaller.

Question: 3.

Partial sums occur when we look at a geometric series $1+z+z^2+z^3+...+z^n$ and consider the following:

$S_1=1$ $\quad$ $S_2=1+z$ $\quad$ $S_3=1+z+z^2$ $\quad$ $S_4=1+z+z^2+z^3$ and so on.

A partial sum can be expressed using sigma notation. For example, $S_4=\sum_{n=0}^{3} z^n$.

a) Use sigma notation on your TI Nspire to find the following sums:

(i) $\sum_{n=0}^{6} w^n$ $\quad$ (ii) $\sum_{n=0}^{8} w^n$ $\quad$ iii) $\sum_{n=0}^{10} w^n$

What do you notice?

(i) 0.875+ i $\quad$ (ii) 1+0.9375i $\quad$ (iii) 1.03125+i

Student responses may vary:

Both real and imaginary parts of all of the partial sums are close to 1, and are becoming increasingly closer to 1.

Download the **Complex Spiral** TNS file to your Software or TI Nspire handheld.

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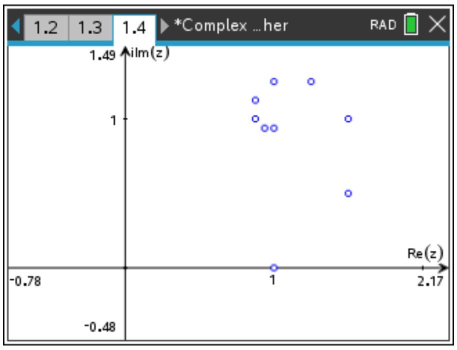
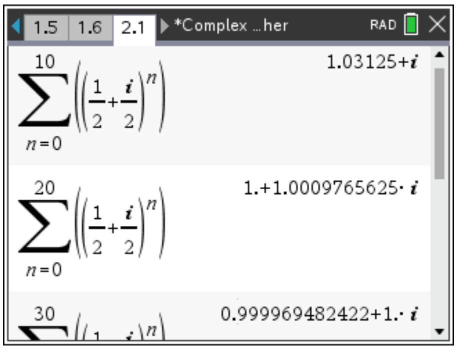
b) Plot the first eight partial sums on the Argand diagram for $w = \frac{1}{2} + \frac{1}{2}i$ and describe the pattern.

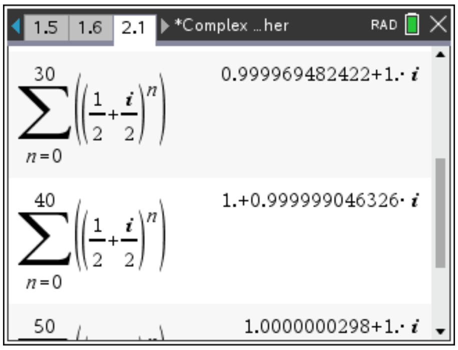
Answer: Student observations may vary. Possible responses may include:

- The points form a spiral pattern (anti clockwise)
- The points are spiralling inwards
- The spiral is spiralling towards the point $1 + i$
- The centre of the spiral looks to be at the point $1 + i$



c) Calculate: $S_{10}, S_{20}, S_{30}, S_{40}, S_{50}$.



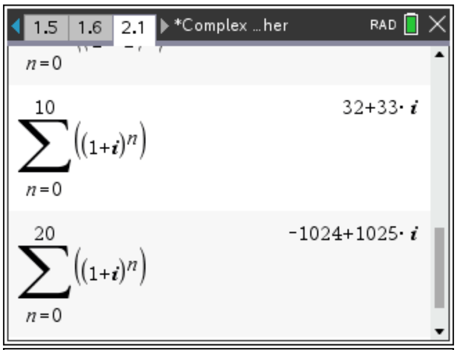


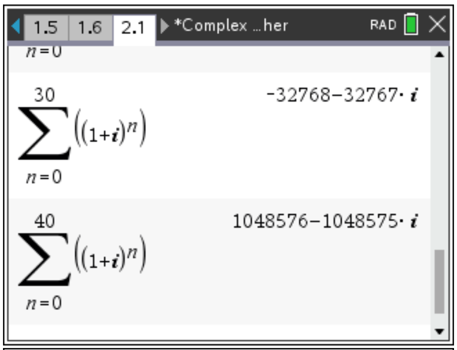
d) What do you notice about the results from part (c)?

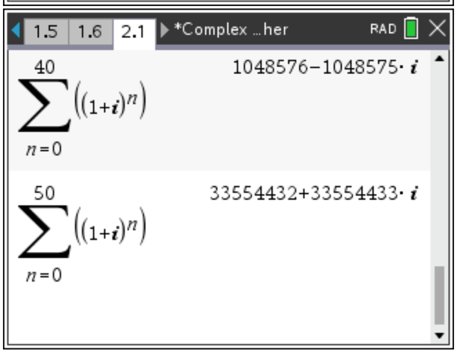
Answer: As the value of n increases, the partial sum approaches: $1+i$

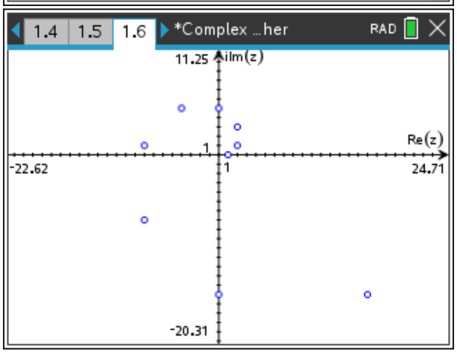
e) Investigate whether a similar pattern occurs for $u = 1 + i$

Answer: It appears that as n becomes larger, the partial sums becomes larger, spiralling outwards.









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f) Repeat questions 2 & 3 for three complex numbers of your choice with the following conditions:

(i) $|z| = 1$ (ii) $|z| > 1.$ (iii) $|z| < 1$

Comment on your findings.

Answer: Student observations may vary. Responses may include:

- For $|z| > 1$, the spirals appear to become larger and spiral outwards. The sequence of powers become larger and larger, and the partials sums become larger.
- For $|z| < 1$, the spirals appear to become smaller and spiral inwards. The sequence of powers becomes smaller, and approaches zero. The partials sums approach a fixed number.
- For $|z| = 1$, the terms repeat themselves in the sequence of powers. Instead of a spiral, you get some points lthat lie on the unit circle. The partial sums do not approach a fixed point, instead they repeat themselves.

Question: 4.

a) Given that $1 + z + z^2 + z^3 + \ldots + z^n$ is a geometric series, determine the conditions when the sum of the series is convergent.

Answer: $|z| < 1$

b) Find the sum to infinity for a convergent $1 + z + z^2 + z^3 + \ldots + z^n$ series.

Answer: $\frac{1}{1-z}$

c) Use your results/observations from questions 2 and 3 to verify your answer to part (b).

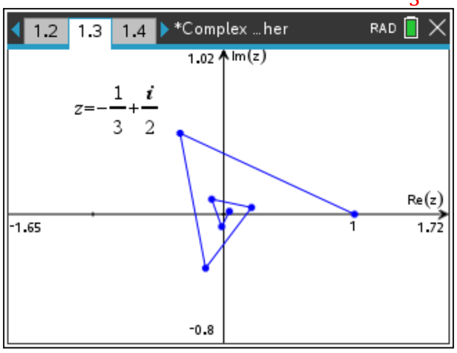
Answers will vary. For the given number in Question 2, $w = \frac{1}{2} + \frac{i}{2}$, $\frac{1}{1-w} = 1 + i$, which is what we observed with the partial sums and the spiral of the partial sums.

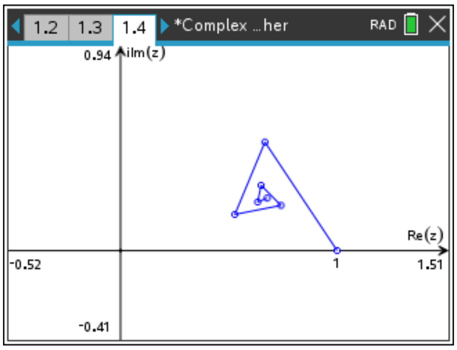
Question: 5.

Select a complex number in each of the four quadrants of the Argand Plane such that $|z| < 1$. Investigate the behaviour of the geometric sequence $1, z, z^2, z^3 \ldots z^n$ and the geometric series $1 + z + z^2 + z^3 + \ldots + z^n$.

a) What do you notice about your results?

Answers will vary. Quadrant 2: $z = -\frac{1}{3} + \frac{i}{2}$.





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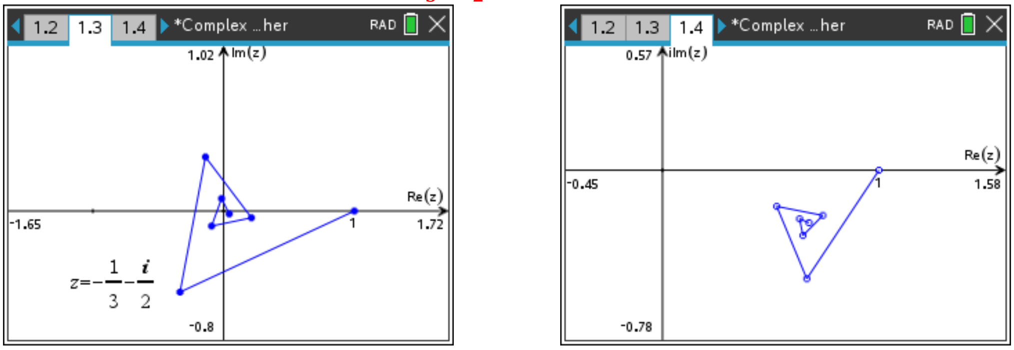
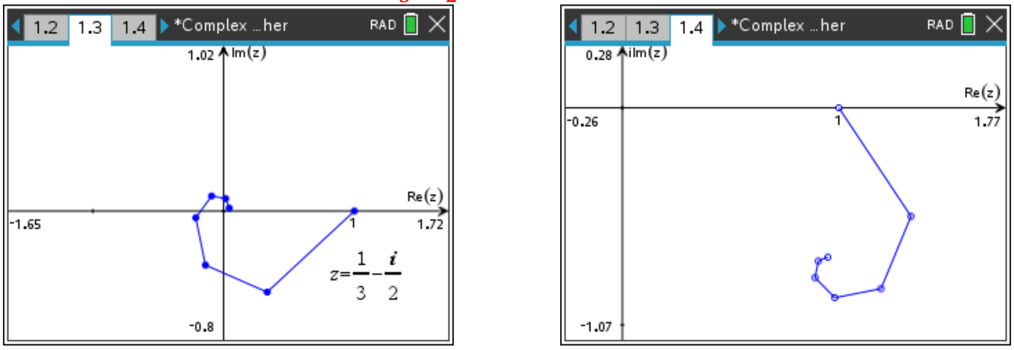
Answer (continued) Quadrant 3: $z = -\frac{1}{3} - \frac{i}{2}$



Answer (continued) Quadrant 4: $z = \frac{1}{3} - \frac{i}{2}$



Answer (continued) Observations may include: The sequence of powers of z^n approaches zero, as long as $|z| < 1$, and the partial sums converge to a particular value

b) Are there any similarities and/or differences between the plots of each of the sequences and corresponding series?

Answer: Student answers will vary. Some observations may include:

- Complex numbers chosen in the 1st and 2nd quadrant cause the spiral to rotate anticlockwise, while a complex number chosen in the 3rd and 4th quadrant will cause the spiral to rotate clockwise.
- Complex numbers chosen in the 1st and 4th quadrant cause the spiral to rotate gradually and spiral gradually towards zero, while complex numbers chosen in the 2nd and 3rd quadrant cause the spiral to rotate quickly and spiral rapidly towards zero

c) What conclusions, if any, could you make?

Answer: Complex numbers chosen in the 1st and 2nd quadrant (with an $Im(z) > 0$) will cause the spiral to rotate anticlockwise, while a complex number chosen in the 3rd and 4th quadrant (with an $Im(z) < 0$) will cause the spiral to rotate clockwise.

Question: 6.

For any complex number $z = x + yi$ it can be shown using DeMoivre's theorem that

$$z^n = (x+yi)^n = |z|^n \cos(n\theta) + |z|^n \sin(n\theta)i.$$

Where $|z|$ is the modulus of z, and $\theta = Arg(z)$ is the (principal) argument of z.

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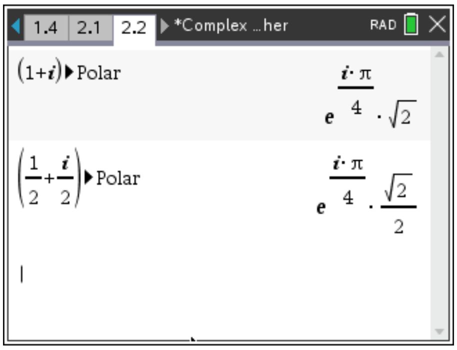
The spiral generated by z^n can be described by the parametric equations:

$$x = |z|^n \cos(n\theta)$$
$$y = |z|^n \sin(n\theta)$$



a) Express z^n in polar form, for:

i. $z = 1 + i$ ii. $z = \frac{1}{2} + \frac{1}{2}i$.

Answer: (i) $\sqrt{2}\,cis\left(\frac{\pi}{4}\right)$ (ii) $\frac{\sqrt{2}}{2}\,cis\left(\frac{\pi}{4}\right)$

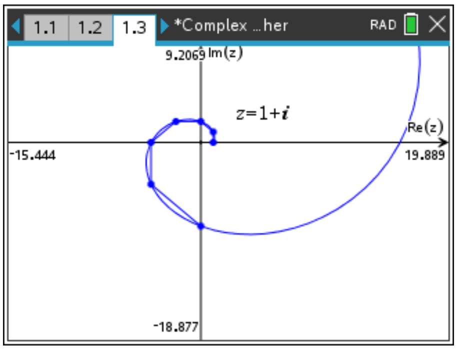
b) Verify that your results in part (a) are correct by calculating z^2, z^3, z^4.

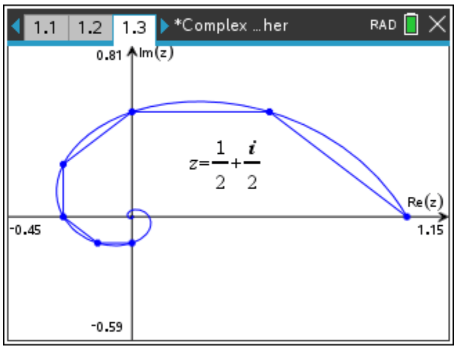
Answers:

(i)
$$z^2 = 2\,cis\left(\frac{\pi}{2}\right) = 2i$$
$$z^3 = 2\sqrt{2}\,cis\left(\frac{3\pi}{4}\right) = -2 + 2i$$
$$z^4 = 4\,cis\left(\frac{3\pi}{2}\right) = -4$$

(ii)
$$z^2 = \frac{1}{2}\,cis\left(\frac{\pi}{2}\right) = \frac{i}{2}$$
$$z^3 = \frac{\sqrt{2}}{4}\,cis\left(\frac{3\pi}{4}\right) = -\frac{1}{4} + \frac{i}{4}$$
$$z^4 = \frac{1}{4}\,cis\left(\frac{3\pi}{2}\right) = -\frac{1}{4}$$

c) Using the TI-Nspire, sketch your parametric equations on a Graph application. How do your results compare with your results from questions 1 and 2?





Answer: Student answers will vary. Some observations may include:

- They are the same spiral
- The spiral passes through the same points as those from Questions 1&2

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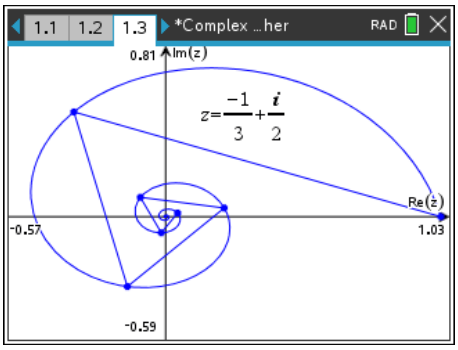
d) Express z^n in terms of cosine and sine for each of the complex numbers that you chose in part 5, and hence sketch the corresponding spirals.

Answers will vary.

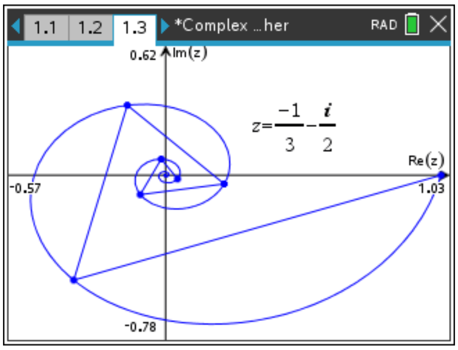
Quadrant 2: $z = -\frac{1}{3} + \frac{i}{2} = \frac{\sqrt{13}}{6}\, cis\left(\pi - \tan^{-1}\left(\frac{3}{2}\right)\right).$

$$z^n = \left(\frac{\sqrt{13}}{6}\right)^n \left[\cos\left(\left(\pi - tan^{-1}\left(\frac{3}{2}\right)\right)n\right) + i\sin\left(\left(\pi - tan^{-1}\left(\frac{3}{2}\right)\right)n\right)\right.$$



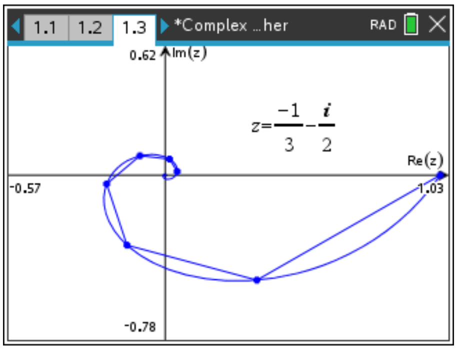
Quadrant 3: $z = -\frac{1}{3} - \frac{i}{2} = \frac{\sqrt{13}}{6}\, cis\left(\pi + \tan^{-1}\left(\frac{3}{2}\right)\right)$

$$z^n = \left(\frac{\sqrt{13}}{6}\right)^n \left[\cos\left(\left(\pi + tan^{-1}\left(\frac{3}{2}\right)\right)n\right) + i\sin\left(\left(\pi + tan^{-1}\left(\frac{3}{2}\right)\right)n\right)\right.$$



Quadrant 4: $z = \frac{1}{3} - \frac{i}{2} = \frac{\sqrt{13}}{6}\, cis\left(-\tan^{-1}\left(\frac{3}{2}\right)\right)$

$$z^n = \left(\frac{\sqrt{13}}{6}\right)^n \left[\cos\left(\left(-tan^{-1}\left(\frac{3}{2}\right)\right)n\right) + i\sin\left(\left(-tan^{-1}\left(\frac{3}{2}\right)\right)n\right)\right.$$



e) What do you notice about your results? Are there any similarities and/or differences?

Answers will vary. Observations may include:

- Complex numbers chosen in the 1st and 2nd quadrant cause the spiral to rotate anticlockwise, while a complex number chosen in the 3rd and 4th quadrant will cause the spiral to rotate clockwise.
- Complex numbers chosen in the 1st and 4th quadrant cause the spiral to rotate gradually and spiral gradually towards zero, while complex numbers chosen in the 2nd and 3rd quadrant cause the spiral to rotate quickly and spiral rapidly towards zero

f) What conclusions, if any, could you make?

Answer: Complex numbers chosen in the 1st and 2nd quadrant (with an $Im(z) > 0$) will cause the spiral to rotate anticlockwise, while a complex number chosen in the 3rd and 4th quadrant (with an $Im(z) < 0$) will cause the spiral to rotate clockwise.

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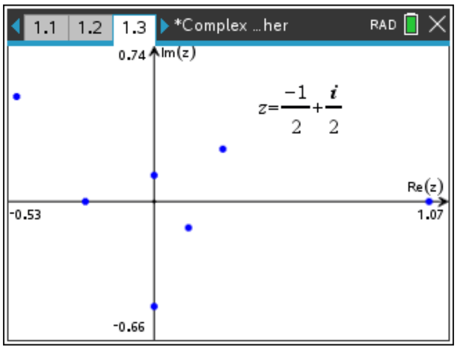
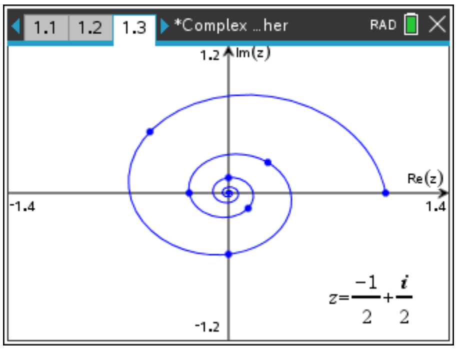
Question: 7 (Extension)

Consider the point $z = -\frac{1}{2} + \frac{1}{2}i.$

(a) Plot the points $1, z, z^2, z^3, z^4, z^5, z^6$ on an Argand Plane

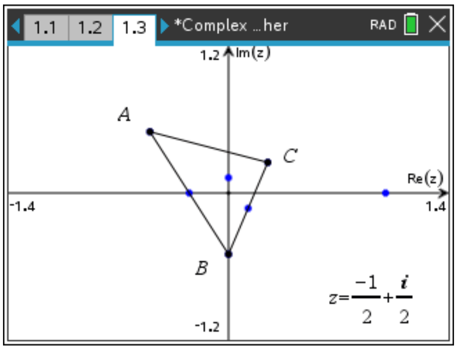


(b) Plot the spiral that is generated by z^n.

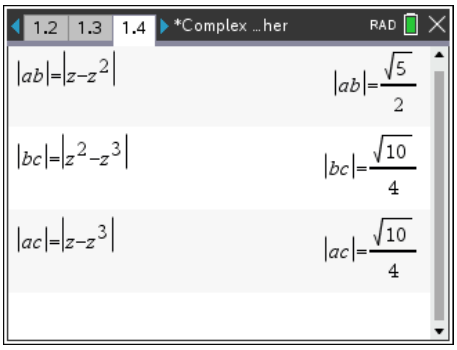


The points z, z^2, z^3 form the vertices of a triangle $\triangle ABC$.

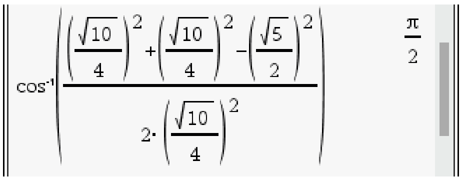
(c) Find the side lengths and interior angles of the triangle $\triangle ABC$.



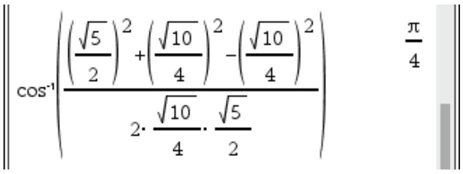
Triangle formed by z, z^2 and z^3.



Side lengths. [Triangle is isosceles since |bc| = |ac|



Angle at C.



Angles at A and B.

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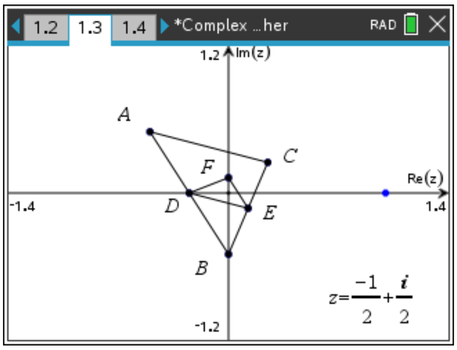
The points z^4, z^5, z^6 form the vertices of another triangle, $\triangle DEF$.

(d) Find the side lengths and interior angles of the triangle $\triangle DEF$. What do you notice?



$$|de|=|z^4-z^5| \qquad |de|=\frac{\sqrt{10}}{8}$$

$$|ef|=|z^5-z^6| \qquad |ef|=\frac{\sqrt{5}}{8}$$

$$|df|=|z^4-z^6| \qquad |df|=\frac{\sqrt{5}}{8}$$

$$\cos^{-1}\left(\frac{\left(\frac{\sqrt{5}}{8}\right)^2+\left(\frac{\sqrt{5}}{8}\right)^2-\left(\frac{\sqrt{10}}{8}\right)^2}{2\cdot\left(\frac{\sqrt{5}}{8}\right)^2}\right) \qquad \frac{\pi}{2}$$

Angle at F

$$\cos^{-1}\left(\frac{\left(\frac{\sqrt{5}}{8}\right)^2+\left(\frac{\sqrt{10}}{8}\right)^2-\left(\frac{\sqrt{5}}{8}\right)^2}{2\cdot\frac{\sqrt{5}}{8}\cdot\frac{\sqrt{10}}{8}}\right) \qquad \frac{\pi}{4}$$

Angles at D and E

The angles in both triangles are the same! The triangles are similar.

The two triangles, $\triangle ABC$ and $\triangle DEF$ are similar.

(e) Find the scale factor, k, such that $|AB| = k|DE|$.

$$\cos^{-1}\left(\frac{\left(\frac{\sqrt{5}}{8}\right)^2+\left(\frac{\sqrt{10}}{8}\right)^2-\left(\frac{\sqrt{5}}{8}\right)^2}{2\cdot\frac{\sqrt{5}}{8}\cdot\frac{\sqrt{10}}{8}}\right) \qquad \frac{\pi}{4}$$

$$\text{solve}\left(\frac{\sqrt{5}}{2}=k\cdot\frac{\sqrt{10}}{8},k\right) \qquad k=2\cdot\sqrt{2}$$

(f) Find the angle that you would rotate $\triangle ABC$ so that it aligns with $\triangle DEF$.

$$Arg(z^6) - Arg(z^3) = \frac{\pi}{4}$$

$$\text{angle}(z^6)-\text{angle}(z^3) \qquad \frac{\pi}{4}$$

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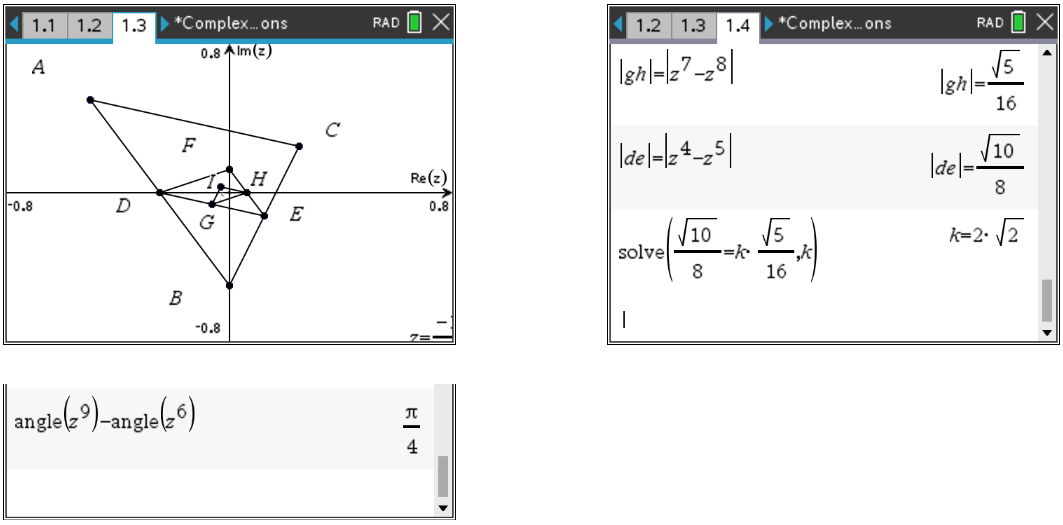
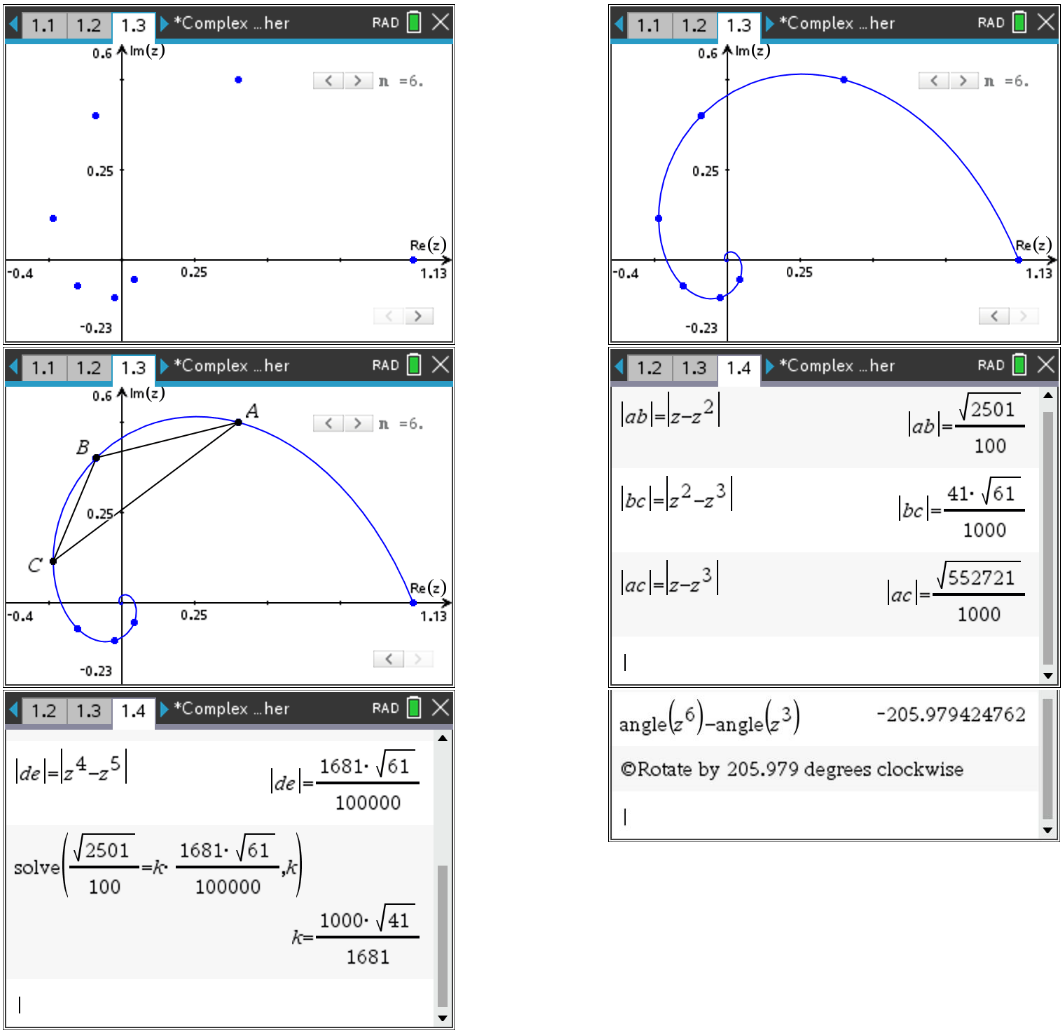
(g) Verify your results from parts (e). and (f) by looking at the triangle created by z^7, z^8, z^9.



(h) Repeat parts (a) – (f) with the point $z = \dfrac{2}{5} + \dfrac{1}{2}i$.



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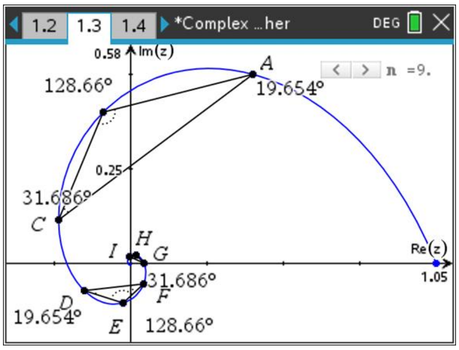

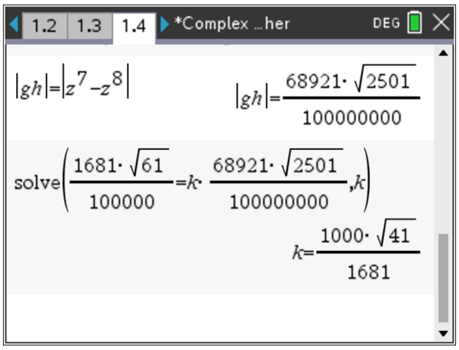

(i) For any complex number, $z = a + bi$ where $a, b \in \mathbb{R}$, find an expression for the scale factor and the angle of rotation that would be needed to be applied to triangle $\triangle ABC$ to obtain $\triangle DEF$.

Answers:

$$|z - z^2| = k\,|z^4 - z^5|$$

$$k = \frac{|z - z^2|}{|z^4 - z^5|}$$

Angle of rotation given by $Arg(z^{n+3}) - Arg(z^n)$

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