Proof of Identity

ID: 9846

Time required
45 minutes

Activity Overview

Students use graphs to verify the reciprocal identities. They then use the handheld's manual graph manipulation feature to discover the negative angle, cofunction, and Pythagorean trigonometric identities. Geometric proofs of these identities are given as well.

Topic: Trigonometric Identities

- *Verify trigonometric identities by graphing.*
- *Use the Pythagorean Theorem to prove the trigonometric identities* $\sin^2\theta + \cos^2\theta = 1$ *and* $\sec^2\theta = 1 + \tan^2\theta$.

Teacher Preparation and Notes

- *This activity is appropriate for an Algebra 2 or Precalculus classroom. The calculator application Cabri Jr. is necessary for completion of this activity.*
- *This activity is intended to be* ***teacher-led*** *with students in* ***small groups****. You should seat your students in pairs so they can work cooperatively on their calculators. You may use the following pages to present the material to the class and encourage discussion. Students will follow along using their calculators.*
- *Students should have experience graphing and translating trigonometric functions.*
- ***To download the Cabri Jr. files and the student worksheet, go to education.ti.com/exchange and enter "9846" in the keyword search box.***

Associated Materials

- *ProofOfIdentity_Student.doc*
- *COFUNCT.8xv (Cabri Jr file)*
- *NEGANGLE.8xv (Cabri Jr file)*

An *identity* is a statement about two expressions that are the same, or *identical*. Trigonometric identities are used to simplify trigonometric expressions and solve trigonometric equations.

Students may be surprised to learn that you already know several trigonometric identities just from knowing the definitions of the trigonometric functions. Here are a few:

$$\csc(x)=\frac{1}{\sin(x)} \qquad \sec(x)=\frac{1}{\cos(x)} \qquad \cot(x)=\frac{1}{\tan(x)}$$

These are called the ***reciprocal identities***. There are 6 reciprocal identities in all.

$$\sin(x)=\frac{1}{\csc(x)} \qquad \cos(x)=\frac{1}{\sec(x)} \qquad \tan(x)=\frac{1}{\cot(x)}$$

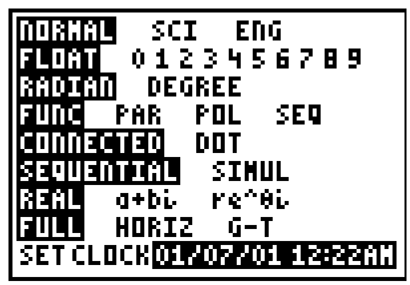
Problem 1 – Negative Angle Identities

If two expressions are the same, what will their graphs look like? If you graph both sides of an identity, their graphs will be exactly the same.

We can use this idea in the other direction to find more trigonometric identities. If two expressions have the same graph, then they are equal.

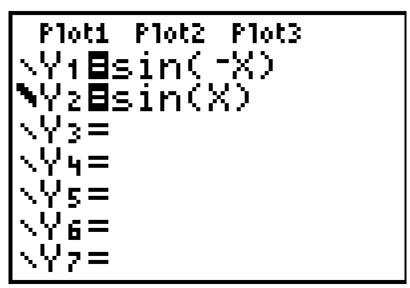
In this problem, students will look for and prove identities that relate trigonometric functions of and angle to trigonometric functions of the opposite angle, such as sin(–*x*) and sin(*x*).

Before beginning, students need to clear any functions from the **Y=** screen and turn off any **StatPlots**. Direct students to press MODE and make sure the calculator is set to measure angles in radians.



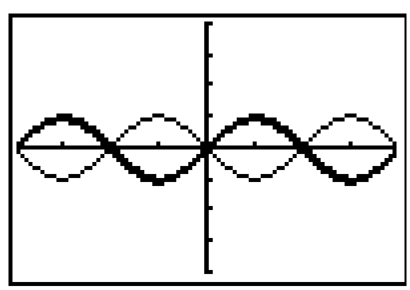
NOTE: Students need to use **1/sin(X)** for cosecant, **1/cos(X)** for secant, and **1/tan(X)** for cotangent.

Press Y=. Enter **sin(–X)** in **Y1** and **sin(X)** in **Y2**. Instruct students to arrow over to the graph type symbol to the left of **Y2** and press ENTER to change it to a thick line. This will help them distinguish between the graphs of **Y1** and **Y2**.



Students need to press ZOOM and select **ZTrig** to set the window size for trigonometric functions.

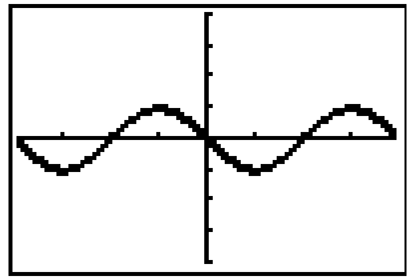
View the graphs. Students are to determine how are they the same and how are they different. Discuss how they can translate the graph of **Y2** (the thick line) to make it line up with **Y1** (the thin line).



Students should conclude that there are two different ways to translate **Y2** to make it line up with **Y1**.

- Flip the graph of **Y2** vertically, reflecting it over the *x*-axis, or
- Shift the graph of **Y2** horizontally

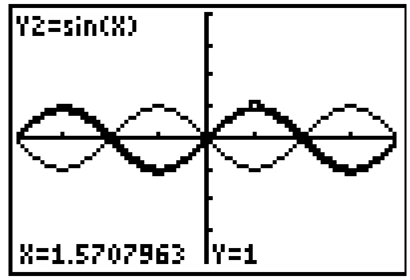
Students are to use what they know about translating graphs to change the equation in **Y2** to reflect it over the *x*-axis. They can check their equation by modifying Y2 in the [Y=] screen and then press [GRAPH], making sure the two graphs align as shown.



Then students are to write the new equation on their worksheet. This is one of the negative angle identities.

sin(–*x*) = **– sin(*x*)**

Students should now change the equation in **Y2** back to **sin(X)**.

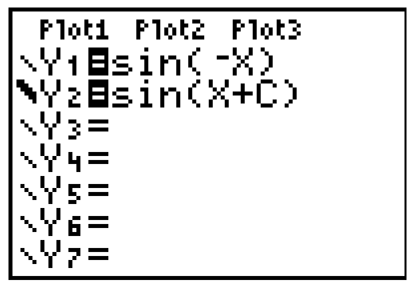


As they trace the graph, they should estimate the horizontal "difference" between the two curves by noting the *x*-values of their peaks.

sin(–*x*) has a peak at *x* = **–1.57** (or 4.71)

sin(*x*) has a peak at *x* = **1.57** (or –4.71)

Students are to use this information and what they know about translating graphs, to change the equation in **Y2** to shift it to the left or right. They should try different values for C until they find the one that makes the two graphs align.

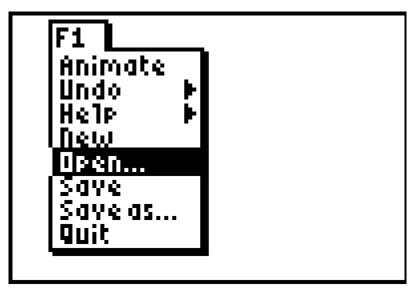


Then students are to write the new equation on the worksheet. This is not a formal trigonometric identity, but it is a true identity that comes from the periodic nature of the sine function.

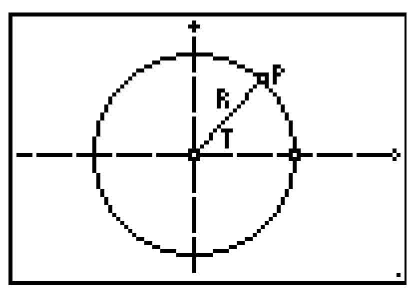
sin(–*x*) = **sin(*x* + π)**

To see a geometric proof of this identity, students need to start the **Cabri Jr.** app by choosing it from the [APPS] menu.

Then open the file named **NEGANGLE**. Press [Y=] to open the **F1: File** menu, then choose **Open…**, then choose **NEGANGLE** from the list.



The file shows the unit circle. A ray intersects the circle at point *P*, creating an angle *T* with the *x*-axis. The segment from the origin to point *P* is labeled *R*.



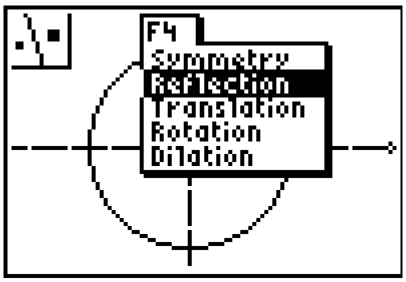
If we let (*x*, *y*) be the coordinates of point *P* and *r* be the length of segment *R*, then we define the three basic trigonometric functions like this:

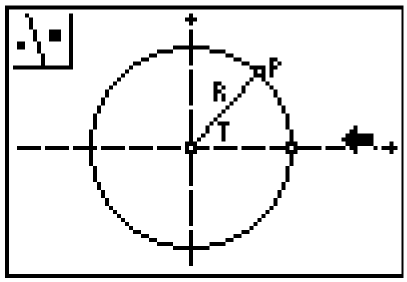
$\sin(T) = \dfrac{y}{r}$ $\cos(T) = \dfrac{x}{r}$ $\tan(T) = \dfrac{y}{x}$

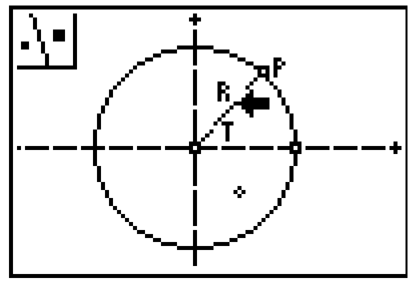
We are interested in the value of these functions for the angle –*T*, the angle with the same magnitude as *T*, but in the opposite direction along the unit circle.

To create this angle by reflecting the segment *R* over the *x*-axis,

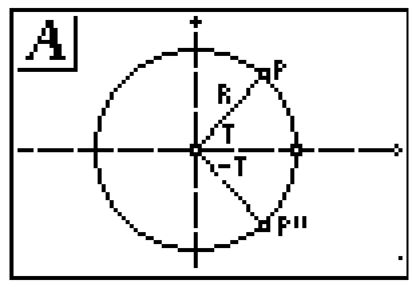
- go to **F4: Transform > Reflection**
- select the line you want to reflect over, the *x*-axis, by moving the cursor to it and pressing ENTER
- select the object you want to reflect, segment R, by moving the cursor to it and pressing ENTER







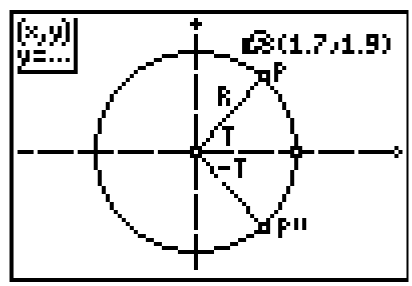
Now students are to use the **Alph-Num** tool (found in the **F5: Appearance** menu) to label the point where the reflected segment intersect the circle **P”**. Label the angle formed by the *x*-axis and the reflected segment –*T*.



Direct students to use the **Coordinates & Equations** tool (found in the **F5: Appearance** menu) to find the coordinates of point *P* and point *P*”.

P is at approximately (1.7, 1.9).

P” is at approximately (1.7, –1.9).

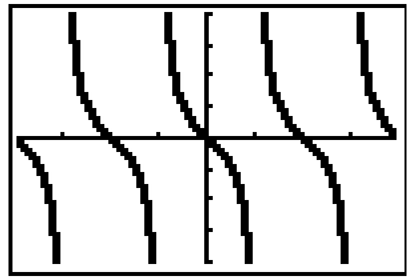


Proving the Negative Angle Identities

1. Write the coordinates of point P” in terms of *x* and *y*, the coordinates of P.
2. Use the coordinates of point P” to write an expression for sin(–*T*).
3. Substitute $\sin(T) = \frac{y}{r}$ in the expression to get sin(–*T*) = –sin(*T*), the negative angle identity you found in the graph! (If you replace *T* with *x*).
4. Repeat these steps to find expressions for cos(–*x*) in terms of cos(*x*) and tan(–*x*) in terms of tan(*x*).

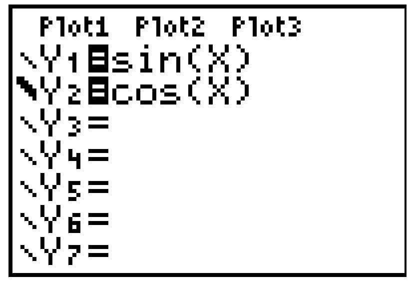
   cos(–*x*) = cos (*x*) tan(–*x*) = –tan (*x*)

Students are to verify these two additional negative angle identities by graphing. Enter the left side in **Y1** and the right side in **Y2**.



Problem 2 – Cofunction Identities

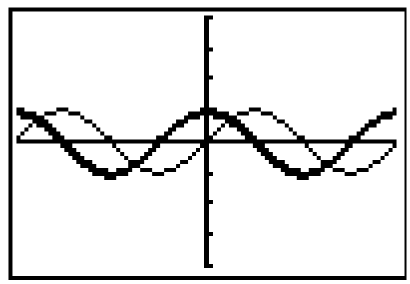
By now, students should have seen enough trigonometric graphs to realize that the sine and cosine functions are more similar to each other than they are to the tangent function. This intuition is formalized by the cofunction identities.



They are to enter **sin(X)** in **Y1** and **cos(X)** in **Y2**.

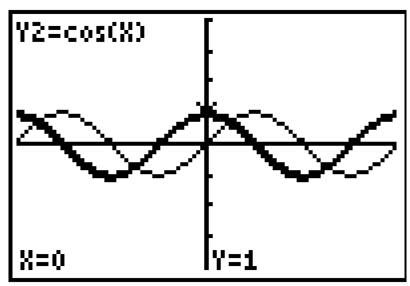
Note: Make sure students arrow over to the graph type symbol to the left of **Y2** and change it to a thick line. This will help distinguish the graphs.

View the graphs. Discuss with students how the graphs are related. How are they the same? How are they different? How can you translate the graph of **Y2** (the thick line) to make it line up with **Y1** (the thin line)?



It looks like you could shift the graph of **Y2** horizontally to make it line up with **Y1**.

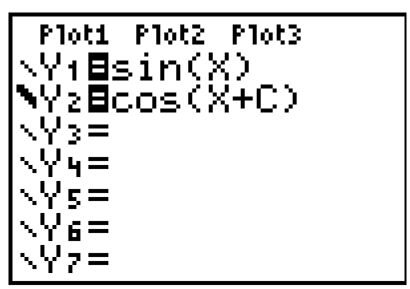
Students are to trace the graph to estimate the horizontal "difference" between the two curves by noting the *x*-value of their peaks.



sin(x) has a peak at x = **1.57** (or –4.71)

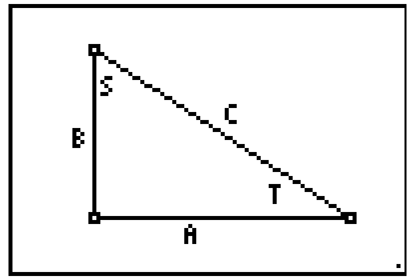
cos(x) has a peak at x = **0** (or –6.28 or 6.28)

They can use this information and what they know about translating graphs to change the equation in **Y2** to shift it to the left or right. Try different values in place of C until you find the one that makes the two graphs align. Remember, C may be negative.



Then, students are to write the new equation on the worksheet. This is called a cofunction identity, because sine and cosine are cofunctions.

$$\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$$

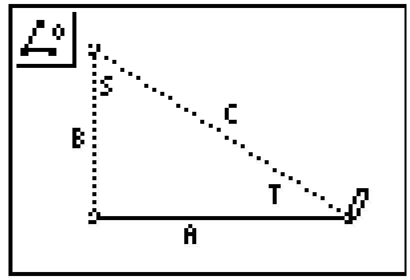


To see a geometric proof of this identity, students need to start the **Cabri Jr.** app by choosing it from the APPS menu.

Open the file named **COFUNCT**. The file shows a right triangle.

Three basic trigonometric functions are defined:

$$\sin(T) = \frac{B}{C} \qquad \cos(T) = \frac{A}{C} \qquad \tan(T) = \frac{B}{A}$$



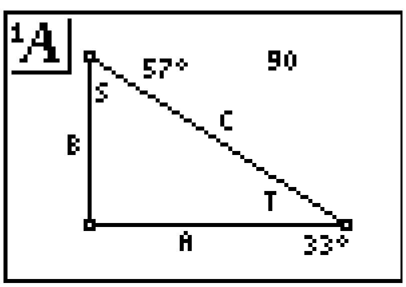
Direct students to use the **Angle Measurement** tool (found in the **F5: Appearance > Measure** menu) to measure angles *S* and *T*. To measure an angle, select a point on the first ray, the vertex, and then a point on the second ray.

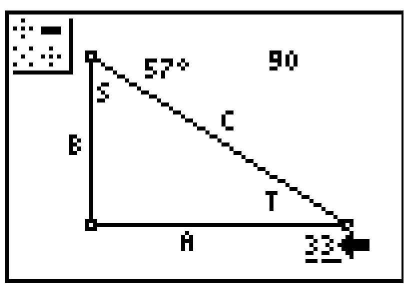
Discuss with students how the two acute angles in a right triangle related. They should use the answer to write an expression for *S* in terms of *T*. (CabriJr measures angles in degrees, so write 90° instead of $\frac{\pi}{2}$.)

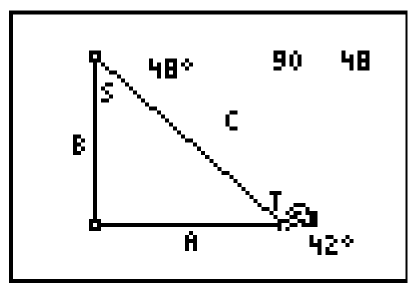
S = **90 – *T***

Students can check their expression with CabriJr.

- Use the **Alpha/Num** tool to type the 90 on the screen. (Press ALPHA to enter numbers.)
- Use the **Calculate** tool (found in the **F5: Appearance** menu) to evaluate the expression for S. To calculate **90 – *T***, select the 90 you typed, then press −, then select the measurement of *T*.
- Drag the vertices of the triangle to check that your expression is true for different values of *S* and *T*.







Proving the Cofunction Angle Identities

1. Use the definition of sine as opposite/hypotenuse to write an expression for the sin(S). Substitute 90 – *T* for *S* and cos(*T*) for $\frac{A}{C}$ to get sin(*T*) = cos (90 – *T*).

2. Substitute *x* for *T* and change degrees to radians to get $\sin(x) = \cos\left(\frac{\pi}{2} - x\right)$.

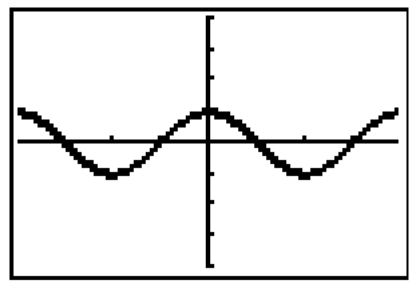
3. Use the negative angle identity to rewrite $\cos\left(\frac{\pi}{2} - x\right)$ as

$$\cos\left(-\left(\frac{\pi}{2} - x\right)\right) = \cos\left(x - \frac{\pi}{2}\right).$$

4. Repeat steps 1 and 2 above to write expressions for cos(*x*) and tan(*x*).

$\cos(x) = \sin\left(x - \frac{\pi}{2}\right)$sin $\tan(x) = \cot\left(x - \frac{\pi}{2}\right)$

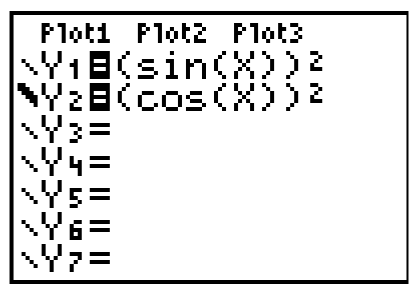
Students can verify these two additional cofunction identities by graphing. Enter the left side in **Y1** and the right side in **Y2**.



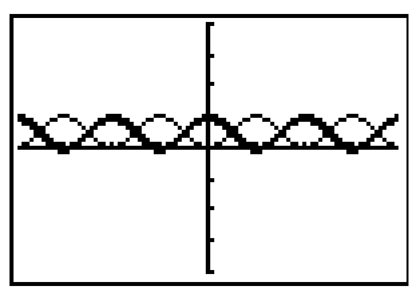
Problem 3 – Pythagorean Identities

The final set of trigonometric identities students will explore in this activity relate the squares of the different trigonometric functions. They are to enter **(sin(X))^2** in **Y1** and **(cos(X))^2** in **Y2**.

Remind students to change the graph type of **Y2** to a thick line.



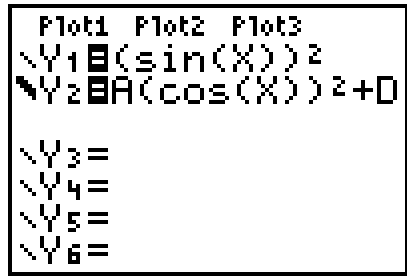
When students view the graphs, they may want to reduce the range of the *y*-axis to zoom in on the graph and get a closer look. Discuss with students how can they translate the graph of **Y2** (the thick line) to make it line up with **Y1** (the thin line).



One way would be to shift the graph of **Y2** horizontally to make it line up with **Y1**. This would be an application of the cofunction identity. Discuss with students how.

$$\sin^2(x) = (\sin(x))^2 = \left(\cos\left(\frac{\pi}{2} - x\right)\right)^2 = \cos^2\left(\frac{\pi}{2} - x\right)$$

The other way would be by flipping the graph, then shifting it vertically. Students should change the equation in **Y2** to flip it and then shift it vertically. They should try different values for in the place of A and D until they find the values that make the two graphs align.

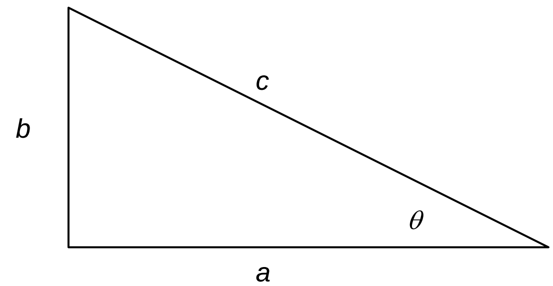


Then, students will write the new equation on the worksheet.

$\sin^2(x)$ = **$1 - \cos^2(x)$**

Proving the Pythagorean Angle Identities

Use the diagram and follow these steps to prove the Pythagorean identities.



1. Write the Pythagorean Formula: $a^2 + b^2 = c^2$.

2. Divide both sides of the Pythagorean Formula by c^2: $\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$

3. Simplify the result. Substitute $\sin\theta$ for $\frac{b}{c}$ and $\cos\theta$ for $\frac{a}{c}$ to yield $\sin^2(x) + \cos^2(x) = 1$.

4. Repeat steps 1 through 3, dividing by a^2 and b^2 to yield additional identities.

- $\sin^2(x) + \cos^2(x) = 1$
- $1 + \cot^2(x) = \csc^2(x)$
- $\tan^2(x) = \sec^2(x) - 1$

Students should verify these two additional Pythagorean identities by graphing. Enter the left side in **Y1** and the right side in **Y2**. Use **1/sin(X)** for cosecant, **1/cos(X)** for secant, and **1/tan(X)** for cotangent.