



## Concepts

A vector-valued function, or vector function, is a function whose domain is the set of real numbers and whose range is a set of vectors. Suppose we consider vector functions  $\mathbf{r}$  whose values are two-dimensional vectors. Then for every number  $t$  in the domain of the vector function  $\mathbf{r}$ , there is a unique vector denoted by  $\mathbf{r}(t)$ .

Let the real-valued functions  $f(t)$  and  $g(t)$  be the component functions of the vector  $\mathbf{r}(t)$ . We often use the notation  $\mathbf{r}(t) = \langle f(t), g(t) \rangle$  to represent the vector function.

Suppose a particle moves in the plane so that at time  $t$  the position of the particle is given by the parametric equations  $x = f(t)$  and  $y = g(t)$ . Let the vector function  $\mathbf{r}$  be defined by  $\mathbf{r}(t) = \langle f(t), g(t) \rangle$ . Then  $\mathbf{r}(t)$  is the position vector of the point  $P(f(t), g(t))$ . Therefore, any continuous vector function  $\mathbf{r}$  describes the position of a particle in two dimensions and, also defines a curve  $C$  traced out by the tip of the moving vector  $\mathbf{r}(t)$ .

Suppose a particle moves along a curve  $C$  so that its position vector at time  $t$  is  $\mathbf{r}(t) = \langle f(t), g(t) \rangle$ .

The derivative of  $\mathbf{r}$  is the velocity vector:  $\mathbf{r}'(t) = \mathbf{v}(t) = \langle f'(t), g'(t) \rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$ .

The speed of the particle at time  $t$  is the magnitude of the velocity vector:

$$\text{speed} = |\mathbf{v}(t)| = |\mathbf{r}'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

The total distance traveled by the particle from time  $t = \alpha$  to time  $t = \beta$  is given by

$$\text{distance traveled} = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2} dt = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The acceleration of the particle at time  $t$  is the derivative of the velocity:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle f''(t), g''(t) \rangle = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle$$

## Course and Exam Description Unit

9.6: Solving Motion Problems' Using Parametric and Vector Valued Functions

## Calculator Files

ParametricMotion.tns

# Exploring Motion in the Plane: Velocity and Acceleration

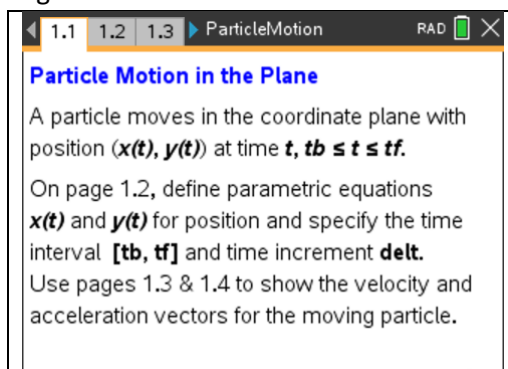
## Using the Document

ParametricMotion.tns: This calculator file provides a technology tool for visualizing the velocity and acceleration vectors at specific times associated with a particle as it moves along a curve  $C$ . The initial curve is defined by

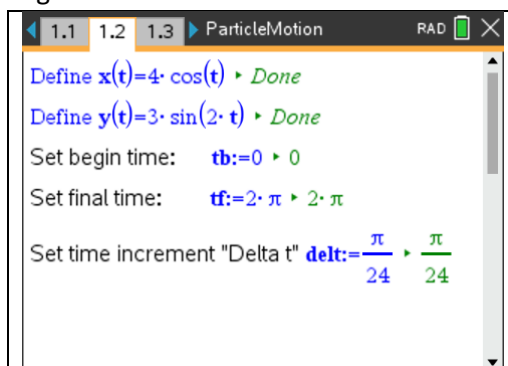
$$x(t) = 4\cos t \quad y(t) = 3\sin(2t)$$

The velocity and acceleration vectors are computed automatically. The complete curve can be displayed or the moving point alone can be displayed (the geometric trace option is used in this case). The velocity and acceleration vectors can be shown or hidden.

### Page 1.1

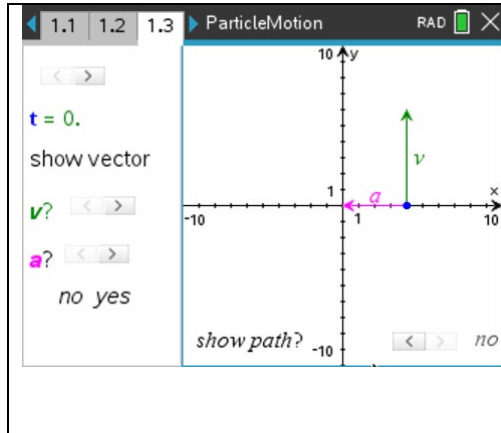
	<p>This introductory screen provides information for defining the parametric equations that describe the motion of a particle in the plane. The functions <math>x(t)</math> and <math>y(t)</math> represent the <math>x</math>- and <math>y</math>- coordinates of the particle at time <math>t</math>. The time interval is specified by <math>[tb, tf]</math>, and the time increment is represented by the variable <math>delt</math>.</p>
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### Page 1.2

	<p>The default settings are <math>x(t) = 4\cos t</math> and <math>y(t) = 3\sin(2t)</math>. The default time interval is <math>[0, 2\pi]</math> and the time increment is <math>\frac{\pi}{24}</math>. All of these values are defined within Math Boxes, and can be changed by the user.</p> <p>Note that there are other calculations at the bottom of this screen. These are used to help construct the graphs on the following screens.</p>
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# Exploring Motion in the Plane: Velocity and Acceleration

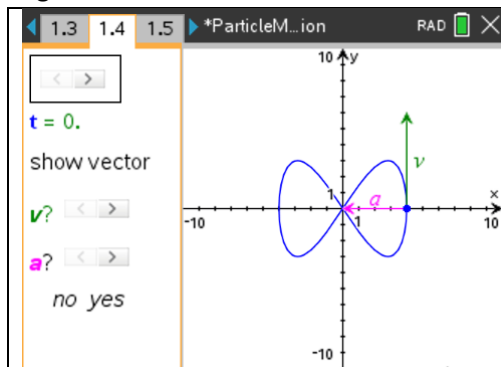
## Page 1.3



The right side of this page shows the point on the curve at time  $t = 0$ , and the velocity and acceleration vectors at that time. The slider at the bottom right of this screen is used to display the entire curve, or path, of the particle, and the points on the curve for each value of  $t$  determined by the increment. Note that the points on the curve are at equal time increments.

On the left screen, the slider for  $t$  is used to move the particle along the curve by the increment. The two sliders below this are used to display or hide the velocity and/or acceleration vectors.

## Page 1.4



This page is similar to Page 1.3. However, on the right screen, the curve is displayed without the specific points on the curve, as determined by the increment.

On the left screen, the slider for  $t$  is used to move the particle along the curve by the increment. The two sliders below this are used to display or hide the velocity and/or acceleration vectors.

## Page 1.5

	A	xpts	B	ypts	C	D
1		4.		0.		
2		3.96578		0.776457		
3		3.8637		1.5		
4		3.69552		2.12132		
5		3.4641		2.59808		
At		=4.				

This Lists and Spreadsheet page is used to construct and plot the points on the curve (Page 1.3) corresponding to each increment of  $t$ . The user does not need to enter any values here nor adjust any formulas.

## Suggested Applications and Extensions

Use the default curve defined by  $x(t) = 4\cos t$ ,  $y(t) = 3\sin(2t)$  to answer the following questions.

Use the slider for  $t$  to observe how the velocity and acceleration vectors change as the particle moves along the curve. Show the entire path and the points on the curve as determined by the increment.

1. How is the velocity vector related to the curve?
2. How is the acceleration vector related to the curve?
3. How can the displayed points be used to explain the speed of the particle?
4. How is the velocity vector related to the speed of the particle?
5. Explain how the velocity and acceleration vectors can be used to predict the next point on the curve, as  $t$  increases.
6. Is the acceleration vector ever perpendicular to the velocity vector? If so, explain the motion of the particle at that time.
7. Find the velocity and acceleration vectors analytically for  $t = \frac{\pi}{4}$ . Display the path, and use the slider for  $t$  to move the particle to the point on the curve where  $t = \frac{\pi}{4}$ . Compare your analytical results with the displayed vectors. Explain any discrepancies.
8. Find an equation of the tangent line to the curve at the point where  $t = \frac{\pi}{6}$ . Explain how the velocity vector of the particle at this time is related to the tangent line.
9. Find all times at which the acceleration vector is  $\langle 0, 0 \rangle$ . Is there any special characteristic of the velocity vector at these times.

Consider the curve  $C$  defined by

$$x(t) = 1 + \int_0^t e^{\cos u} du \quad y(t) = 2\sin t$$

For  $0 \leq t \leq \pi$ .

1. Find the velocity and acceleration vectors at time  $t = \frac{\pi}{4}$ . Confirm your results on the graph. Explain why the velocity and acceleration vectors are pointing in the opposite directions.
2. Use the graph to estimate the time at which the speed of the particle is greatest. Describe the velocity and acceleration vectors at that time.
3. Find, analytically, the time at which the particle is at its highest point on the curve. Find the  $x$ - and  $y$ -coordinates at that time. Describe the velocity and acceleration vectors at that time, and explain why these are consistent with this position on the curve.
4. Use Page 1.5 to find the straight line distance between the points on the curve that correspond to  $t = 0$  and  $t = \frac{\pi}{4}$ . Find the length of the curve between  $t = 0$  and  $t = \frac{\pi}{4}$ . Compare these two values and explain why they are close in magnitude.
5. Challenge: Describe the concavity of this curve for  $0 \leq t \leq \pi$ . Do you think there is a relationship between the velocity and acceleration vectors that can determine the concavity of the curve? If so, describe this relationship.

# Exploring Motion in the Plane: Velocity and Acceleration

For each of the following curves, describe the velocity and acceleration vectors as  $t$  increases through the given interval.

1.  $x(t) = \cos^3 t$      $y(t) = \sin^3 t$      $0 \leq t \leq 2\pi$
2.  $x(t) = 2\sqrt{t} - 8$      $y(t) = t^2 - 3t - 3$      $0 \leq t \leq 4$
3.  $x(t) = 5e^{-t}$      $y(t) = \cos t$      $0 \leq t \leq 2\pi$
4.  $x(t) = \ln(t + 1)$      $y(t) = \frac{4t}{t^2 + 1}$      $0 \leq t \leq 10$
5.  $x(t) = \sin(2t)$      $y(t) = \tan^{-1} t$      $0 \leq t \leq 2\pi$

## Extended Application Questions

1. For a given curve, turn off the Show Path, and turn off Show Vector **a** (the acceleration vector). Use the velocity vector to predict the motion of the particle. Explain your reasoning.
2. For a given curve, turn off the Show Path, and display only the velocity and acceleration vectors. Use these two vectors to predict the motion of the particle. Explain your reasoning.
3. Explain how the displayed points can be used to determine the relative speed of the particle.
4. Explore various curve and try to explain how the velocity and acceleration vectors can be used to determine the concavity of the curve at a point.