Teacher Notes

Activity 18

Exploring Infinite Series

Objectives

- Identify a geometric series
- Determine convergence and sum of geometric series
- Identify a series that satisfies the alternating series test
- Use a graphing handheld to approximate the sum of a series

Materials

- TI-84 Plus / TI-83 Plus

Teaching Time

- 90 minutes

Abstract

Geometric series are analyzed in detail. The first part of the activity could serve as students' first introduction to series. The graphing handheld is used as a tool to investigate partial sums of series in both tabular and graphical representations. The alternating series test and error bound are introduced in the second part of the activity.

Management Tips and Hints

Prerequisites

Students will benefit from familiarity with the idea of a sequence and the use of summation notation.

Student Engagement

This activity can be used very early in the introduction of series in a calculus course.

If students already have been introduced to geometric sequences and series, then they could begin with the graphing handheld part of the activity.

Evidence of Learning

Given a series, students should be able to identify

- whether it is geometric.
- whether a geometric series converges.
- the sum of convergent geometric series.
- when a series satisfies the conditions of the alternating series test.
- the error bound to determine how many terms of a partial sum of a convergent alternating series are needed to achieve a predetermined level of accuracy in approximating the sum of the series.

Common Student Errors/Misconceptions

Students may confuse convergence of the individual terms of a series with the convergence of the series itself. Students often overlook the monotonically decreasing term requirements as part of the alternating series test.

Teacher Notes

In the **Y=** editor in sequence mode, it is not necessary to fill in the first term of the sequence *u*(*n*min) if the formula for *u*(*n*) is explicitly in terms of *n*. If you use the sequence mode for investigating recursive sequences (where *u*(*n*) is defined in terms of preceding terms), then this initial value may be required.

Extensions

The language used with the geometric series emphasizes the word *ratio* in the analysis of convergence. This is deliberately done to provide a transition to the *ratio test* for convergence, if the teacher follows with that topic. Indeed, the *ratio test* is based on making a comparison with a suitable geometric series:

If

$$\sum_{n=1}^{\infty} a_n$$

is a series such that

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

then if $L < 1$, the series converges. If $L > 1$ or $L = \infty$, then the series diverges. If $L = 1$, the test is inconclusive.

Note that there is not an error bound associated with the ratio test, though.

In Questions **6** through **12**, some of the convergent alternating series are based on classic power series and thus can actually be evaluated. These examples could be interesting to revisit after students have worked with power series. For your reference, these are:

6. $\ln\left(\frac{3}{2}\right) \approx 0.4054651081$

7. $e^{-1} \approx 0.3678794412$

8. $\sin\left(\frac{1}{2}\right) \approx 0.4794355386$

9. $\cos\left(\frac{1}{3}\right) \approx 0.9449569463$

11. $\text{atan}(1) = \frac{\pi}{4} \approx 0.7853981634$

12. $\frac{1/2}{(1+1/2)^2} = \frac{1/2}{9/4} = \frac{2}{9} \approx 0.222222222$

Activity Solutions

1. $\sum_{n=0}^{\infty} 5 \cdot \left(\frac{^{-}3}{4}\right)^n$

Appropriate **Y=** and **WINDOW** settings are shown below.

```
 Plot1  Plot2  Plot3
 nMin=1
:u(n)=sum(seq(5*
(-3/4)^K,K,0,n)
 u(nMin)={5}
:v(n)=
 v(nMin)=
:w(n)=
```

```
WINDOW
 nMin=0
 nMax=10
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=10
↓Xscl=1
```

```
WINDOW
↑PlotStep=1
 Xmin=0
 Xmax=10
 Xscl=1
 Ymin=-.1
 Ymax=6.1
 Yscl=1
```

The first 11 terms in decimal form are:

5
$^{-}3.75$
2.8125
$^{-}2.109375$
1.58203125
$^{-}1.186523438$
0.8898925781
$^{-}0.6674194336$
0.5005645752
$^{-}0.3754234314$
0.2815675735

In fraction form:

5

$\frac{^{-}15}{4}$

$\frac{45}{16}$

$\frac{^{-}135}{64}$

$\frac{405}{256}$

$\frac{^{-}1215}{1024}$

$\frac{3645}{4096}$

$\frac{^{-}10935}{16384}$

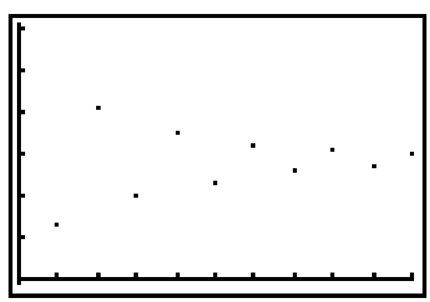
$\frac{32805}{65536}$

$\frac{^{-}98415}{262144}$

$\frac{295245}{1048576}$

The partial sum is ≈ 2.977814674.

The plot of the partial sums for $n = 0$ to $n = 10$ is



2. $\sum_{n=0}^{\infty} {}^{-}2 \cdot \left(\frac{5}{3}\right)^n$

The first 11 terms in decimal form are:

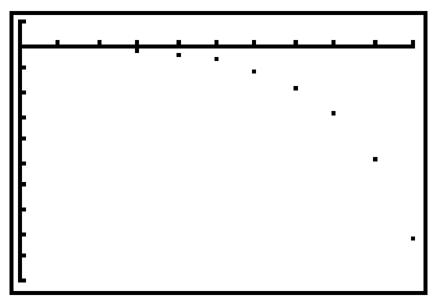
⁻2	⁻42.86694102
⁻3.333333333	⁻71.44490169
⁻5.555555556	⁻119.0748362
⁻9.259259259	⁻198.4580603
⁻15.43209877	⁻330.7634338
⁻25.72016461	

In fraction form:

⁻2	$\frac{-31250}{729}$
$\frac{-10}{3}$	$\frac{-156250}{2187}$
$\frac{-50}{9}$	$\frac{-781250}{6561}$
$\frac{-250}{27}$	$\frac{-3906250}{19683}$
$\frac{-1250}{81}$	$\frac{-19531250}{59049}$
$\frac{-6250}{243}$	

The partial sum is $\approx$ ⁻823.908584 .

The plot of the partial sums for n = 0 to n = 10 (with Ymin = ⁻1000, Ymax = 100, and Yscl = 100) is



3. The series is a geometric series because the ratio of each term to the one preceding it is a constant $r = \frac{-1}{3}$.

It converges because $-1 < r < 1$, and with a first term of $a = \frac{3}{2}$, the sum is

$$\frac{a}{1-r} = \frac{(3/2)}{1-(-1/3)} = \frac{3/2}{4/3} = \frac{3}{2} \cdot \frac{3}{4} = \frac{9}{8}$$

The first 11 terms in decimal form are:

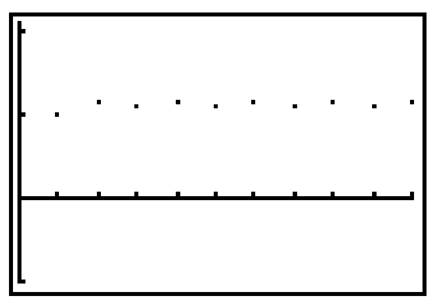
1.5	0.0020576132
-0.5	-6.858710562 E -4
0.1666666667	2.286236854 E -4
-0.0555555556	-7.620789514 E -5
0.0185185185	2.540263171 E -5
-0.0061728395	

In fraction form:

$\frac{3}{2}$	$\frac{1}{486}$
$\frac{-1}{2}$	$\frac{-1}{1458}$
$\frac{1}{6}$	$\frac{1}{4374}$
$\frac{-1}{18}$	$\frac{-1}{13122}$
$\frac{1}{54}$	$\frac{1}{39366}$
$\frac{-1}{162}$	

The partial sum for $n = 10$ is ≈ 1.125057156.

The plot of the partial sums for $n = 0$ to $n = 10$ (with Ymin = -1, Ymax = 2.1, and Yscl = 1) is



4. The series is a geometric series because the ratio of each term to the one preceding it is a constant

$r = \frac{3}{2}$.

It does not converge because $r > 1$.

The first 11 terms in decimal form are:

0.0069444444
0.0104166667
0.015625
0.0234375
0.03515625
0.052734375
0.0791015625
0.1186523438
0.1779785156
0.2669677734
0.4004516602

In fraction form:

$\frac{1}{144}$

$\frac{1}{96}$

$\frac{1}{64}$

$\frac{3}{128}$

$\frac{9}{256}$

$\frac{27}{512}$

$\frac{81}{1024}$

$\frac{243}{2048}$

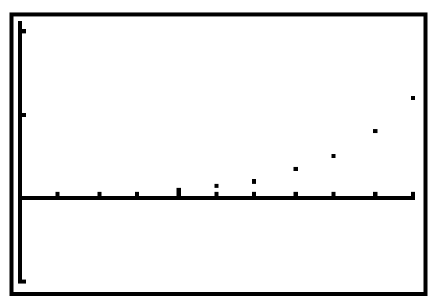
$\frac{729}{4096}$

$\frac{^-2187}{8192}$

$\frac{6561}{16384}$

The partial sum for $n = 10$ is ≈ 1.187466092.

The plot of the partial sums for $n = 0$ to $n = 10$ (with Ymin = $^-1$, Ymax = 2.1, and Yscl = 1) is



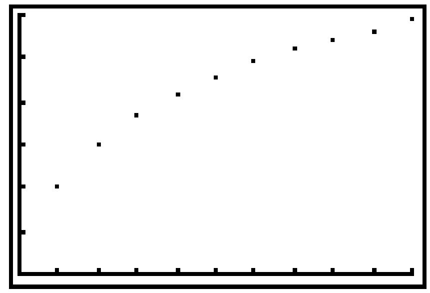
5. The series is *not* a geometric series because the ratio of each term to the one preceding it is not constant. For example, the ratio of the second term to the first term is

$$\frac{1/2}{1} = \frac{1}{2}$$

but the ratio of the third term to the second term is

$$\frac{1/3}{1/2} = \frac{2}{3}$$

Some students may believe that this series converges because its terms clearly approach 0, but the harmonic series diverges. The partial sum for $n = 10$ is 2.92896825420, the partial sum for $n = 20$ is 3.597739657, and the partial sum for $n = 50$ is 4.499205338. The plot of the partial sums up to $n = 100$ takes quite a while to graph, but is as shown:



6. $$\sum_{n=1}^{\infty} \frac{(^-1)^{n+1}}{n \cdot 2^n} = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \ldots + \frac{(^-1)^{n+1}}{n \cdot 2^n} + \ldots$$

This series meets the requirements of the alternating series test because the terms alternate in sign and approach 0 monotonically. If the absolute value of the term represented by

$$|a_{N+1}| = \frac{1}{(N+1) \cdot 2^{N+1}} < 0.01$$

then the *Nth* partial sum is within 0.01 of the actual sum. The smallest value of N for which this is true is $N = 4$, and that partial sum is approximately equal to 0.4010416667.

7. $$\sum_{n=1}^{\infty} \frac{(^-1)^{n+1}}{(n+1)!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots + \frac{(^-1)^{n+1}}{(n+1)!} + \ldots$$

This series meets the requirements of the alternating series test because the terms alternate in sign and approach 0 monotonically. If the absolute value of the term represented by

$$|a_{N+1}| = \frac{1}{(N+2)!} < 0.01$$

then the *Nth* partial sum is within 0.01 of the actual sum. The smallest value of N for which this is true is $N = 3$, and that partial sum is equal to $\frac{3}{8} = 0.375$.

8. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)!} = \frac{1}{2} - \frac{1}{2^3 \cdot 3!} + \frac{1}{2^5 \cdot 5!} + \ldots + \frac{(-1)^n}{2^{2n+1}(2n+1)!} + \ldots$

This series meets the requirements of the alternating series test because the terms alternate in sign and approach 0 monotonically. If the absolute value of the term represented by

$$|a_{N+1}| = \frac{1}{2^{2N+3}(2N+3)!} < 0.01$$

then the *Nth* partial sum is within 0.01 of the actual sum. The smallest value of *N* for which this is true is *N* = 1, and that partial sum is approximately equal to 0.4791666667.

9. $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n}(2n)!} = 1 - \frac{1}{3^2 \cdot 2!} + \frac{1}{3^4 \cdot 4!} + \ldots + \frac{(-1)^n}{3^{2n}(2n)!} + \ldots$

This series meets the requirements of the alternating series test because the terms alternate in sign and approach 0 monotonically. If the absolute value of the term represented by

$$|a_{N+1}| = \frac{1}{3^{2N+2}(2N+2)!} < 0.01$$

then the *Nth* partial sum is within 0.01 of the actual sum. The smallest value of *N* for which this is true is *N* = 1, and that partial sum is equal to $\frac{17}{18}$ or approximately 0.9444444444.

10. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n+1} = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \ldots + \frac{(-1)^{n+1} n}{(n+1)} + \ldots$

This series does *not* meet the requirements of the alternating series test because the terms do not approach 0.

11. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{(-1)^n}{(2n+1)} + \ldots$

This series meets the requirements of the alternating series test because the terms alternate in sign and approach 0 monotonically. If the absolute value of the term represented by

$$|a_{N+1}| = \frac{1}{2N+3} < 0.01$$

then the *Nth* partial sum is within 0.01 of the actual sum. The smallest value of *N* for which this is true is *N* = 49, and that partial sum is approximately equal to 0.7803986631.

12. $$\sum_{n=1}^{\infty} \frac{(^-1)^{n+1} n}{2^n} = \frac{1}{2} - \frac{2}{4} + \frac{3}{8} - \frac{4}{16} + \ldots + \frac{(^-1)^{n+1} n}{2^n} + \ldots$$

This series meets the requirements of the alternating series test because the terms alternate in sign and approach 0 monotonically. If the absolute value of the term represented by

$$|a_{N+1}| = \frac{N+1}{2^{N+1}} < 0.01$$

then the *Nth* partial sum is within 0.01 of the actual sum. The smallest value of *N* for which this is true is $N = 9$, and that partial sum is equal to 0.228515625.