Exploration 2

Teacher Notes

Exploration: Greatest Common Divisor & Least Common Multiple

Learning outcomes addressed

1.7 Factor any pair of integers to find their gcd and lcm.

1.8 Use the commands *gcd* and *lcm* to verify the gcd and lcm of any pair of integers.

Lesson Context

Finding the greatest common divisor (gcd) and least common multiple (lcm) of two integers is an important skill for calculating the common denominator to add or subtract fractions. For this reason, the learning outcomes listed above are included in virtually all state guidelines.

Recently, biologists have discovered that the lcm of the life cycles of an insect species and its predator plays a vital role in that species' survival. By adjusting its life cycle so it has no factor in common with the life cycle of its main predator, an insect species can increase its chances of survival. As researcher Glenn Webb asserted, "The prime-number life cycle is no coincidence, but has evolved as an effort to avoid predators." In the life cycle of the cicadas (see the discussion in the lesson included here) we see a dramatic application of the lcm in the natural world.

Lesson Launch

Have students read the section on the cicada. Ask initiating questions such as:

- Why is it an advantage for a species to have a different life cycle from its main predator?
- In what years is a species is most threatened if the life cycle of the species is 3 years and the life cycle of its predator is 6 years?

  answer:Every 6 years after the initial coincidence.
- If a species and its predator both appear in the same year, will they ever appear together in the same year?

  answer: Yes. If p and q are the life cycles, coincidence will occur every lcm(p, q) years.

Lesson Closure

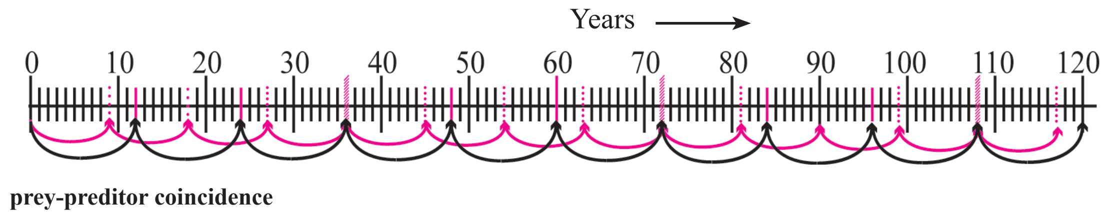
Ensure that students understand that the gcd(p, q) is the *largest* integer that is a divisor of both p and q. Furthermore lcm(p, q) is the *smallest* integer that is a multiple of p and q. Ensure also that students know how to factor two numbers (with and without technology) to find their gcd and lcm. From the TI-*n*spire Investigation in the *Exercises* students should discover that gcd(p, q) × lcm(p, q) = $p \times q$.

Student Work Sheet

Exploration: Greatest Common Divisor & Least Common Multiple

❶. Using the number line below, list the years that are multiples of 12.

❷. List the years that are multiples of 9.

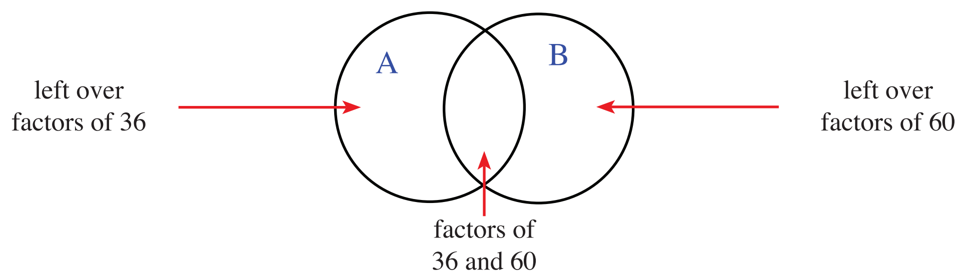


❸. What years are multiples of both 9 and 12?

❹. What is the smallest integer that is a multiple of both 9 and 12?

❺. Use the number line above to find the smallest integer that is a multiple of 9 and 13.

❻. Express 36 and 60 as products of the powers of prime numbers.
In the intersection of the two circles, list all the factors that are common to 36 and 60 as often as they occur. Calculate the product of these factors.



❼. List in region A the factors of 36 that are left over and in region B, the factors of 60 that are left over. Calculate the product of the factors in region A and multiply by the product of the factors in region B (but do not include the factors in the intersection).

❽. In the calculator application of TI-*n*spire, calculate gcd(36, 60) and lcm(36, 60). Compare with your answers in ❻. and ❼.

❾. Use TI-nspire to calculate gcd(75712, 61516) and lcm(75712, 61516).

TI-nspire Investigation

Follow the instructions in the TI-*n*spire Investigation in the *Exercises.*
Then complete this statement:

If p and q are two positive integers, then gcd(p, q) × lcm(p, q) =

Exploration 2: Greatest Common Divisor & Least Common Multiple

Every 17 years in parts of the United States, plagues of the chirping cicadas fill the trees, only to disappear a few weeks later. Since 1634, the cicada plagues have recurred every 17 years up to the most recent plague in 2008. Why does the cicada have a life cycle of 17 years?

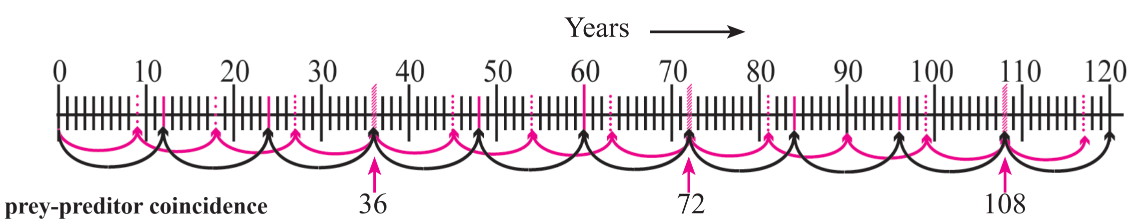
Scientists were provided with a clue when they discovered that another species of cicada has a life cycle of 13 years. They conjectured that emerging only once in a prime number of years enabled the cicada to elude the cicada-killer wasp or praying mantis. If the life cycle of the cicada were 12 years, its emergence would always coincide with that of a predator having a life cycle of 2, 3, or 4 years. Using prey-predator models, biologists showed that any species with predators and the ability to change its life cycle through mutation will eventually develop a life cycle that is a prime number.

Example 1

A particular species of insect has a life cycle of 12 years and its predator has a life cycle of 9 years. Eventually both the prey and predator will emerge in the same year. How many years before the two species coincide again?

Solution

We draw a number line and mark 0 as a point of coincidence. The predator emerges every 9 years after the coincidence, that is, at 9 years and all the multiples of 9. The red arrows show the years when the predator emerges.



The prey appears every 12 years after the first coincidence, that is at 12 years and all the multiples of 12. The black arrows show the years when the prey emerges.

The next coincidence occurs at a multiple of 12 years (for the prey) and a multiple of 9 years (for the predator), i.e., the smallest multiple of 9 and 12. This is called the *least common multiple* (lcm) of 9 and 12. Comparing the two sets of multiples, we see that the lcm of 9 and 12 is 36.

multiples of 12: 12, 24, 36, 48, 60, …

multiples of 9: 9, 18, 27, 36, 45, 54, …

Therefore the two species coincide every 36 years, i.e., after 36, 72, 108, … years.

Worked Examples

Example 2

Suppose the prey described in *Example 1* changes its life cycle from 12 to 13 years. How many years are there between successive coincidences of prey and predator if the predator's life cycle remains at 9 years?

Solution

The predator appears on the multiples of 9: 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, 108, and 117.
The prey appears on the multiples of 13: 13, 26, 39, 52, 65, 78, 91, 104, and 117.

Comparing the multiples of 9 and 13, we can see that the lcm of 9 and 13 is 9×13 or 117. Therefore, successive coincidences occur every 117 years. By changing its life cycle from 12 to 13 years, the cicada has given itself a huge survival advantage. The predators need to learn about least common multiples! For your survival, we review the definitions of the greatest common divisor and least common multiple.

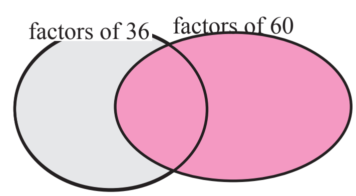
Definitions: The *greatest common divisor* of two positive integers *a* and *b* [denoted gcd(*a, b*)] is the largest integer that divides them both.

The *least common multiple* of two positive integers *a* and *b* [denoted lcm(*a, b*)] is the smallest positive integer that is a multiple of both integers.

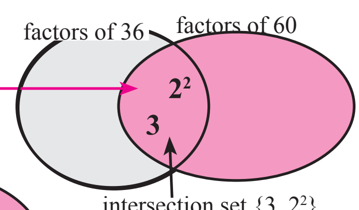
Example 3

Find the gcd and lcm of 36 and 60

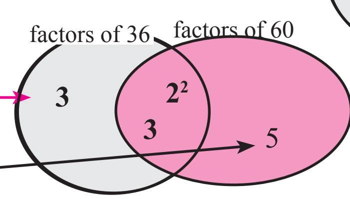
Solution



Step 1 Create a Venn diagram in which to place the factors of 36 and 60.
Factor 36 and 60: $36 = 2^2 \cdot 3^2$ and $60 = 2^2 \cdot 3 \cdot 5$.



Step 2 List the prime power factors that are common to 36 and 60 and write them in the intersection set of a Venn Diagram.



Step 3 Write the remaining prime factors of 36 outside the intersection set.

Step 4 Write the remaining prime power factors of 60 outside the intersection set.

The gcd(36, 60) is the product of the factors in the intersection set, i.e., $2^2 \cdot 3$ or 12.
The lcm(36, 60) is the product of all the factors in the diagram, i.e., $3 \cdot 2^2 \cdot 3 \cdot 5$ or 180.

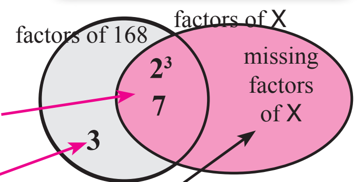
Example 4

Riddle: I am the year X in which John F. Kennedy became President of the United States. What year am I?

Clue: gcd(168, X) = 56 and lcm(168, X) = 5880.

X represents an unknown number

Solution



Proceeding as in Example 3, we create a Venn diagram for the factors of 168 and X. Then we write the prime factors of the gcd(168, X), i.e., 56 in the intersection set.

Next, we write the remaining factors of 168 (i.e., 3) outside the intersection set.

The missing factors of X times the other factors in the Venn Diagram must equal 5880. That is, 168 times the missing factors of X = 5880. So the missing factors of X are equal to 5880 ÷ 168 or 35.

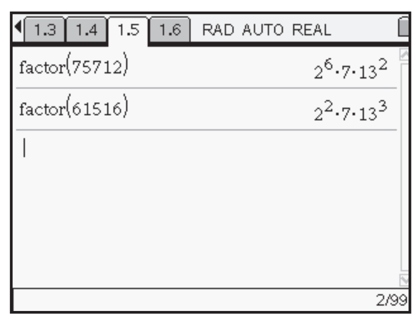
Then X = $2^3 \cdot 7 \cdot 35$ or 1960. John F. Kennedy became President in 1960.

The following example shows how to use TI-Nspire to check your computation of the gcd or lcm or to find the gcd or lcm of a pair of large numbers.

Example 5

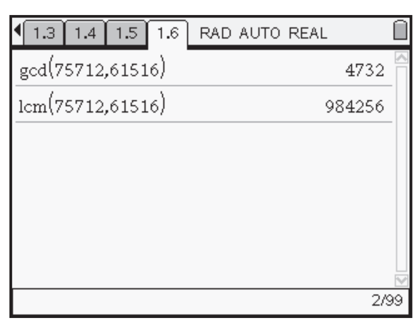
Find the gcd and lcm of 75712 and 61516.

Solution



We access the *factor* command by pressing the menu key and choosing **Number** > **Factor** > enter . This is the same as: menu 2 2 .

As shown in the display, we obtain 75712 = $2^6 \cdot 7 \cdot 13^2$ and 61516 = $2^2 \cdot 7 \cdot 13^3$. Proceeding as in *Example 3*, we write, gcd(75712, 61516) = $2^2 \cdot 7 \cdot 13^2$ or 4732, and lcm (75712, 61516) = $2^6 \cdot 7 \cdot 13^3$ or 984256.



To verify these answers, we can use the commands **gcd(** and **lcm(**. These commands can be typed directly from the keypad, accessed from the catalog, or accessed from the menu as follows:

Press the menu key and then choose **Number** > **Greatest Common Divisor**. Enter the two numbers separated by a comma and press enter. For the least common multiple, press menu >**Number** > **Least Common Multiple**.

The screen display verifies the values for the gcd and lcm found above.

Exercises and Investigations

❶. Evaluate.
a) gcd(26,39) b) gcd(56, 98) c) gcd(44, 121)

❷. Evaluate.
a) lcm(26, 39) b) lcm(56, 98) c) lcm(44, 121)

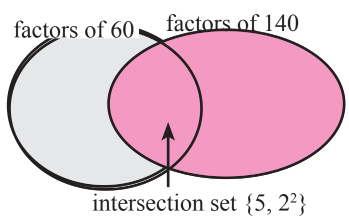
❸. Evaluate.
a) gcd(51,323) b) gcd(873,546) c) gcd(18267,18269)

❹. Evaluate.
a) lcm(51, 323) b) lcm(873, 546) c) lcm(18267, 18269)

❺. Two comets, *Alpha* and *Beta* have periodic orbits with periods of exactly 76 years and 48 years respectively. When both comets appear in the same year they are said to be in *conjunction*. How many years are between successive conjunctions?

❻. Find the unknown number X such that gcd(24, X) = 8 and lcm(24, X) = 48.

❼. To find the gcd and lcm of 60 and 140, draw a Venn diagram like the one shown here. Factor each number into its prime powers. Write the common factors in the intersection region. Write the remaining factors of 60 and 140 outside the intersection. Multiply the factors in the intersection set to get the gcd of 60 and 140. Multiply all the factors in the Venn diagram to get the lcm of 60 and 140.



❽. Construct a Venn Diagram as in Exercise ❼ to find two numbers with gcd = 5 • 7 • 13 and lcm = $5^2 \cdot 7^2 \cdot 13$. Can you find two other numbers that have the same gcd and lcm?

❾. Find two numbers which have gcd 374 and lcm 12,716. Is there more than one answer? If so, list another pair of such numbers. (Hint: Factor 374 and 12,716 and place their factors in a Venn diagram as in Example 4.)

❿. Riddle: I am the year X in which Ronald Reagan was elected President for his first term. What year am I? **Clue:** gcd(756, X) = 36 and lcm(756, X) = 41,580.

⓫. a) What is the gcd of two prime numbers?
b) What is the lcm of two prime numbers?
c) Use part b) to explain why it is a survival advantage for an insect to have a life cycle that is a prime number of years.

⓬. **An interesting property of the gcd of two numbers**
Try this recipe.

Step 1 Write down two positive integers such that neither is a multiple of the other. Call the larger number x and the smaller number y.

Step 2 Divide x by y and record the remainder. Call it r.

Step 3 Calculate gcd(x, y) and gcd(y, r). What do you discover?

Step 4 Repeat steps 1 through 3 with two different numbers. Conjecture a theorem. Test your conjecture. When you learn algebra, you will be able to prove this theorem.

⓭. The TI-Nspire command **remain(31,10)** gives the remainder when 31 is divided by 10. To access the **remain**(command, type it from the keyboard, use the catalog or press these keys:

menu 2 5

When a number is divided by 10, the remainder is its last digit.
a) Calculate **remain(7^2, 10)** to find the last digit of 7^3.
b) Find the last digits of 7^3, 7^4, 7^5, 7^6, 7^7, 7^8, 7^9 and 7^{10}.
c) Conjecture the last digit of 7^{11}. Check your conjecture.
d) Conjecture the last digit of 7^{144}? Check your conjecture.

⓮. The TI-Nspire command **remain(x, y)** gives the remainder when a positive integer x is divided by smaller positive integer y. Choose two 3-digit numbers. Call the larger number x and the smaller y. Follow the recipe in Exercise 12 to find gcd(x, y) and gcd(y, r). If $r \neq 0$, repeat to obtain gcd(r, **remain**(y, r)). Repeat until you get a 0 remainder. What do you discover?

TI-nspire Investigation

Download the file titled *LCM & GCD* to obtain a spreadsheet. Enter an integer in column A and another in column B. Column C displays their gcd. Column D displays their lcm. Column E displays the product of the numbers in columns C and D.

Enter 6 small integers in column A and another 6 in column B. Compare columns A and B with column E.

a) Conjecture a relationship between the numbers in columns A and B and the number in column E.

b) Enter larger integers into columns A and B and compare with column E to test your conjecture.

Answers to the Exercises & Hints for the Investigations

Exploration 2

❶ a) 13 b) 14 c) 11

❷ a) 78 b) 392 c) 484

❸

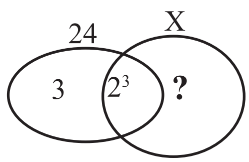
1.3	1.4	1.5	1.6 RAD AUTO REAL
gcd(51,323)			17
gcd(873,546)			3
gcd(18267,18269)			1

❹

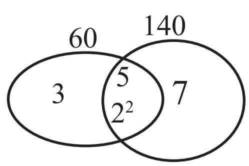
1.4	1.5	1.6	1.7 RAD AUTO REAL
lcm(51,323)			969
lcm(873,546)			158886
lcm(18267,18269)			333719823

❺ Successive conjunctions occur every lcm(48, 76) years, i.e., every 912 years.

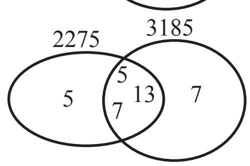
❻ The factors in the intersection set are the factors of 8, i.e., 2^3. Since $24 = 2^3 \times 3$, then the factor of 24 outside the intersection set is 3. Since lcm is 48 or $2^4 \times 3$, then the factor of X that is outside the intersection set is a factor of 2. That is, $X = 2^3 \times 2$ or 16.



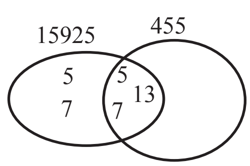
❼ The gcd is the product of the factors in the intersection set, i.e., 20. The lcm is the product of all the factors in the Venn diagram, i.e., $3 \times 5 \times 2^2 \times 7$ or 420.



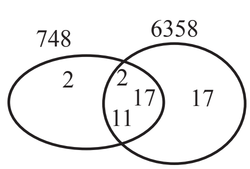
❽ The gcd is the product of the factors in the intersection set, i.e., $5 \times 7 \times 13$. The remaining factors are 5 and 7. If we place one in each set outside the intersection, we obtain the numbers $5^2 \times 7 \times 13$ or 2275 and $7^2 \times 5 \times 13$ or 3185.



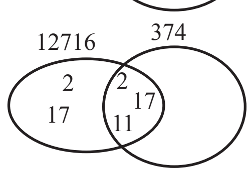
If we place both of the remaining factors, 5 and 7 in the same set, we obtain the numbers $5^2 \times 7^2 \times 13$ or 15925 and $5 \times 7 \times 13$ or 455.



❾ The gcd is the product of the factors in the intersection set, i.e., $2 \times 11 \times 17$. The remaining factors are 2 and 17. If we place one in each set outside the intersection, we obtain the numbers $2^2 \times 11 \times 17$ or 748 and $17^2 \times 2 \times 11$ or 6358.



If we place both of the remaining factors, 2 and 17 in the same set, we obtain the numbers $2^2 \times 17^2 \times 11$ or 12716 and $2 \times 11 \times 17$ or 374.



⓾ **Riddle:** The first year of Reagan's first term was 1980.

⑪ a) The gcd of two prime numbers is 1 because they share no common factors except 1.
b) The lcm of two prime numbers is their product.
c) If an insect and its predator have life cycles that are different prime numbers *p* and *q*, they will coincide every $p \times q$ years. If their life cycles share a large common factor, they will coincide more frequently.

Exploration 2 cont'd

⑫ This property is the basis of the Euclidean algorithm for finding the gcd of two numbers. Since gcd(*x*, *y*) = gcd(*x*, *r*), we can apply the division algorithm on successively smaller numbers until we reach the final remainder which is equal to gcd(*x*, *y*).

⑬ The final digit of powers of 7 cycles through the numbers: 7 ➔ 9 ➔ 3 ➔ 1.
Examine the last four digits of 7^{145}, 7^{245} and 7^{345}. What do you see? Explain why this happens.

⑭ See the answer to *Exercise 12* above.

Hint for the TI-nspire Investigation

Can you explain why the numbers in Column E are the product of the numbers in Columns C and D?