

Rational Thinking

Teacher Notes



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TI-Nspire CAS



Investigation



Student



45min

Aim

Is it possible to find a number that when multiplied by itself will equal 20?

The main focus of this activity is to establish an understanding of the concept of irrational numbers.

Students however come from a 'rational' perspective, that is there must be a number, it can't go on forever.

The activity also focuses on the understanding of the square root of a number and what is meant as a perfect square.

Equipment

For this activity you will need:

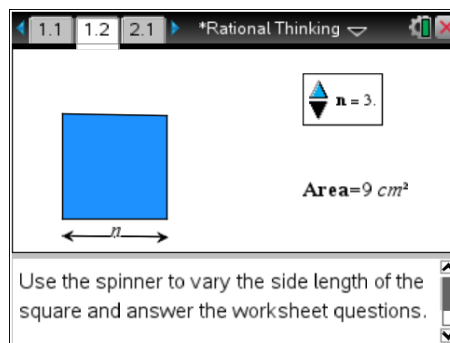
- TI-Nspire CAS
- TI-Nspire CAS file (tns): Rational Thinking

Rational Investigation:

Open the TI-nspire document "Rational Thinking".

Navigate to page 1.2

Move the mouse over the spinner and use it to change the side length and subsequent area of the square.



1. Adjust the side length of the square (using the spinner) to obtain the following areas:

- Area = 25cm^2 Side length = **5cm**
- Area = 16cm^2 Side length = **4cm**
- Area = 9cm^2 Side length = **3cm**

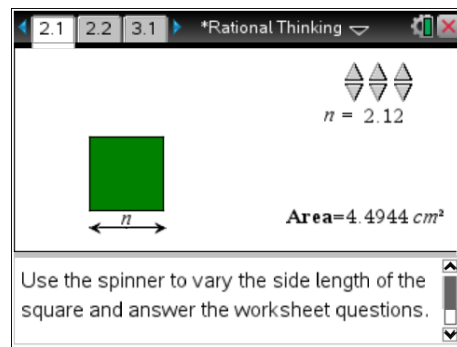
Perfect squares are selected to begin with and the spinner can only produce whole numbers.

2. If a square has an area of 20cm^2 the side length would need to be more than **4** cm and less than **5** cm.

This reasoning should be based on 4^2 is too small and 5^2 is too big. These answers are used as feedback to provide a starting point in the search for the 'answer'.

Navigate to problem 2.1

On this page three spinners are used to adjust the side length of the square. The first spinner changes the units, the second changes tenths and the third changes hundredths.



The spinner on this page provides for decimal values. It is worthy of note to observe the students if they adjust the smaller valued spinners to zero and then start with the units spinner. With access to the TI-Navigator system, this part of the investigation is worthy of a class screen capture and then highlight whether students were systematically exploring or more random.

3. Adjust the spinners to make the area of the square as close to 20cm^2 as possible.
 - a. What is the side length that produces an area as close as possible to 20cm^2 ? **4.47cm**
 - b. A square with area 20cm^2 has a side length between **4.47cm** and **4.48cm**.

All students should get the same answers here based on 4.47 being too small and 4.48 being too big.

4. Adjust the spinners to make the area of the square as close to 10cm^2 as possible.
 - a. What is the side length that produces an area as close as possible to 10cm^2 ? **3.16cm**
 - b. A square with area 10cm^2 has a side length between **3.16cm** and **3.17cm**.
5. Adjust the spinners to make the area of the square as close to 8cm^2 as possible.
 - a. What is the side length that produces an area as close as possible to 8cm^2 ? **2.83cm**
 - b. A square with area 8cm^2 has a side length between **2.82cm** and **2.83cm**.
 - c. How many spinners do you think would be required to create the exact side length for a square with area 8cm^2 ? **Infinitely many... or its not possible. (Discussion point)**

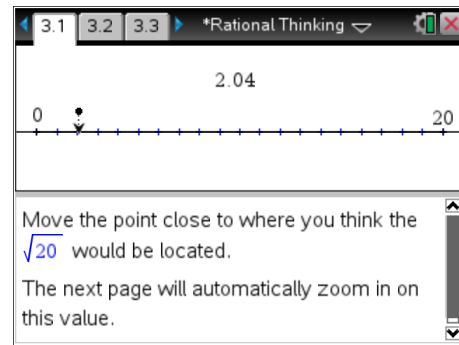
*This is a good discussion point and where the concept that the number is 'irrational'. The thought that a number does not terminate is very challenging conceptually. Note that if students do understand, this point is worthy of a 'maths joke'. "A student has taught himself pi to 30 decimal places and decides to try and trick his teacher. **Student:** Do you want to play a game where we take it in turns to recall the digits of pi. **Teacher:** Sure... you can go first, but lets start with the least significant digit."*

Navigate to problem 3.1

On this page there is a single point that can be moved along the number line. The aim is to move the point as close to $\sqrt{20}$ as possible.

Due to the scale, it may not be possible to get the point really close; however the next page zooms in on your estimate allowing you to make a better approximation.

Adjust the position of the point then answer the next question.



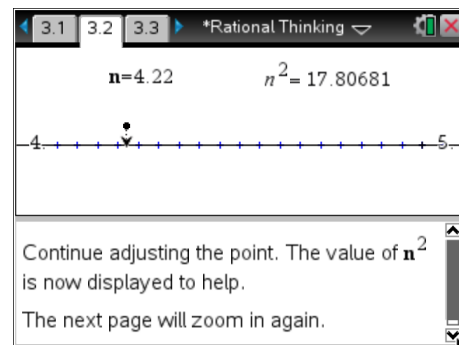
6. My estimate that I made for $\sqrt{20}$ using this moveable point was **answers will vary $\approx 4.47\text{cm}$** .

If the TI-Navigator system is available, it is worth capturing the screen of each student's calculator when estimating square roots using this number line approach. Often students place the initial guess half way between 0 and 20, these students have generally confused square root with a halving process.

Navigate to problem 3.2

This page zooms in on your approximation for $\sqrt{20}$.

Move the point closer to your estimate of $\sqrt{20}$, this time the computed value of your estimate is displayed to provide some assistance.

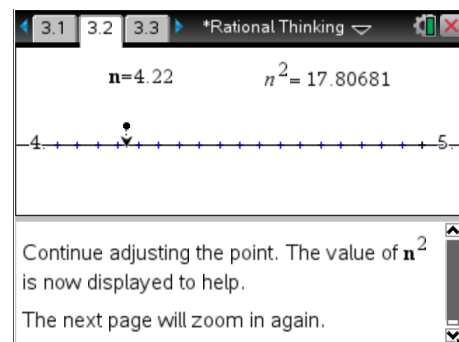


7. The $\sqrt{20}$ lies between: **4.0cm and 5.0cm. (Answers will vary – these values should be the first set of zoomed points).**

Navigate to problem 3.3

This page zooms in even closer to your approximation for $\sqrt{20}$.

Move the point closer to your estimate, notice that n^2 is displayed with more decimal places as your approximation improves the accuracy of the answer must also improve.



8. My estimate for $\sqrt{20}$ is: **4.47214cm**.
 9. My estimate squared is: **20.000036cm²**.

Note that rounding from the number line for the estimate and the squared result can make small differences to the actual results. Discussion at this point of the investigation relates to whether or not it would be possible to determine a number, when squared, would produce 20.00000... (zeros for “ever”), exactly 20. It seems ‘rational’ enough, but the square root of 20 is ‘irrational’.

Babylonian Investigation:

In the previous investigation feedback for the estimate was provided by seeing the estimate squared. The estimate could be increased or decreased accordingly. The Babylonian method uses a different method for feedback. To understand how this feedback works, consider the following:

$$\sqrt{81} = 9$$

This means: $9 \times 9 = 81$

Therefore $\frac{81}{9} = 9$

This concept can be used if the square root of the number is not known:

Estimate $\sqrt{20} \approx 4$ Guess (4)

But $\frac{20}{4} = 5$ Feedback (5)

Therefore $\sqrt{20} \neq 4$ But the answer must lie between the guess: 4 and the feedback: 5.

A reasonable step forward would be to average the guess and the feedback:

New Estimate: $\frac{4+5}{2} = 4.5$ New Guess (4.5)

But $\frac{20}{4.5} = 4.4$ Feedback (4.444...)

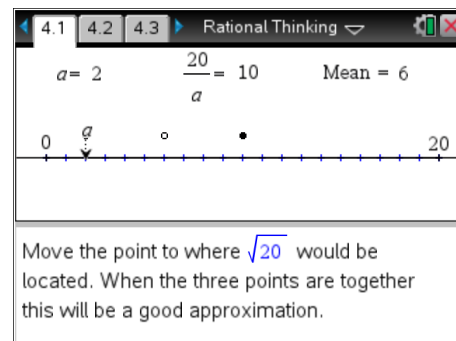
As this process is repeated the approximation improves.

Using the teacher software the next visual is worth repeating as it illustrates why the Babylonian method provides such a good approximation after relatively few iterations, particularly in comparison to a guess and check method which generally focuses on improving the accuracy of the next significant digit. The first iteration is shown by the ‘hollow’ point. Place a mark at this point then move the solid point to this location and watch how rapidly the three points merge. This process underpins many other iterative processes used by computers.

Navigate to page 4.1

Move point a and observe how the other two points move. The feedback is the bold point. The average is the hollow point.

When all three points are together this will be a good approximation.



10. My first estimate using the Babylonian method for $\sqrt{20}$ is: 4.47cm.

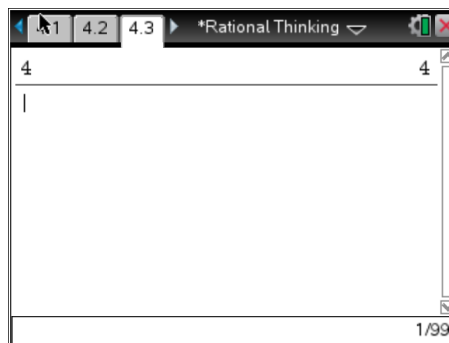
Notice that even on this scale the answer is accurate to two decimal places.

The next stage of the investigation moves from the visual representation to purely numeric. The two dimensional display of the calculation, its retention on the screen and the automatic replacement of the term 'ans' with the actual value assists in understanding this process.

Navigate to page 4.3

For illustrative purposes suppose $\sqrt{20}$ is estimated as 4.

Enter the number 4 and press [Enter]

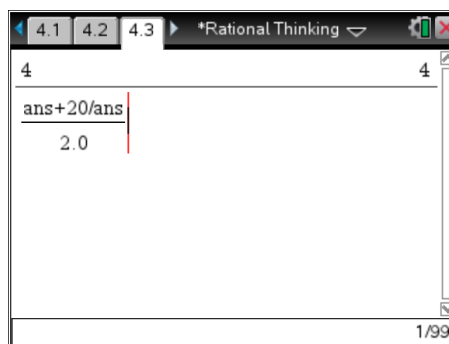


Feedback can be generated using the following:

$$\frac{ans+20/ans}{2.0}$$

Note: For *ans* Press CTRL + (-)

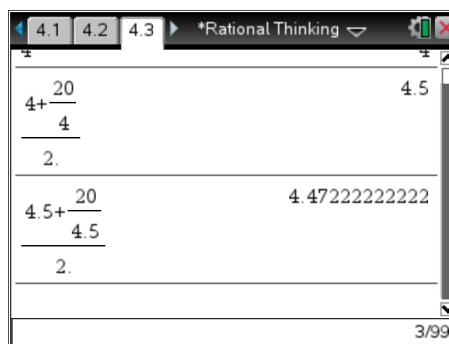
Use 2.0 for the denominator as this will produce a decimal answer when the calculator is in the 'automatic' calculation mode.



Press [Enter] twice (as shown opposite) and notice how the reference to **ans** is replaced with the actual value.

The original estimate for $\sqrt{20}$ is still just visible at the top of the screen. The first feedback answer is 4.5 followed by 4.4722...

This second feedback response is a relatively close approximation as $4.7222^2 \approx 20.0008$



11. Determine the approximate value for the $\sqrt{20}$ after three (3) iterations of this process.

4.472135956 – Notice the remarkable accuracy after just three (3) iterations.

12. When the feedback answer ceases to change, the value has been computed to the capacity of the calculator. How much iteration was required to compute the approximate value for $\sqrt{20}$?

Four iterations (4.4721359549996) – Students should appreciate how such a simple process can get such an 'accurate' estimate... however students should be reminded that this process would continue for 'ever' and still not arrive at a single answer. The number system is '**continuous**' and that a part of this continuity is the set of **irrational** numbers.

13. Use the Babylonian method for computing approximate values for the square root of the following numbers. Include your original estimate and a justification for your estimate, the approximate answer (8 decimal places) and the number of iterations required to obtain the answer.

a. $\sqrt{60}$ Estimate: 7cm (Between 7 & 8). Answer ≈ 7.74596692 (4 iterations)

b. $\sqrt{46}$ Estimate: 6cm (Between 6 & 7). Answer ≈ 6.78232998 (5 iterations)

c. $\sqrt{29}$ Estimate: 5cm (Between 5 & 6). Answer ≈ 5.38516481 (5 iterations)

d. $\sqrt{80}$ Estimate: 8cm (Between 8 & 9). Answer ≈ 8.94427191 (4 iterations)

14. Do you think it is possible to calculate the exact value for $\sqrt{20}$?

This is the most significant question. Whilst the investigation focuses on finding the square root of a single number (20), it should be highlighted that 2, 3, 5, 6, 7... have the same issue. The notion of a continuous number system where a continued halving or averaging process will never reach a steady state for such calculations necessitates a system for dealing with 'irrational numbers'. This leads directly into the next investigation: "Irrational Thoughts".