EXPLORATIONS™

Activity 5

Objectives

- Associate a meaning for right, left, and symmetric difference quotients that include rate of change and graphical interpretations
- Use symmetric difference quotients to approximate instantaneous rate of change

Materials

- TI-84 Plus / TI-83 Plus

Average Rate of Change, Difference Quotients, and Approximate Instantaneous Rate of Change

Introduction

In algebra and precalculus courses, it is likely that you studied functions thoroughly. You found values of functions at given input values, input values that yielded roots of functions, and input values at the intersection of two functions. Each of these tasks involved identifying particular function input and output values or, from a graphical perspective, finding particular points on the graphs of the functions.

In calculus, understanding how function values are changing as input values change is more important than finding the values of a function at specific input values. A first step in this direction is exploring the concept of the average rate of change of a function between two input values.

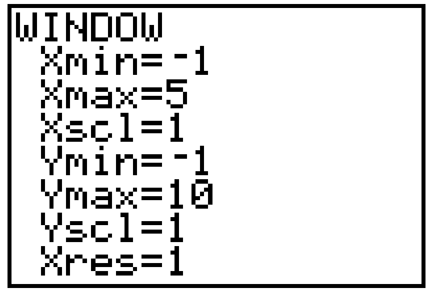
Exploration

The *average rate of change* of a function f between two function input values, x_1 and x_2, is given by

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

1. Find the average rate of change of $f(x) = x^2$ between $x_1 = 1$ and $x_2 = 3$.

The average rate of change between two points is the slope of the line connecting the two points.



2. Use algebraic reasoning to find the slope-intercept equation of the line through the point (1, 1) with a slope equal to the rate of change from your answer to Question **1**.

This line is called the *secant line* of the function f between points x_1 and x_2.

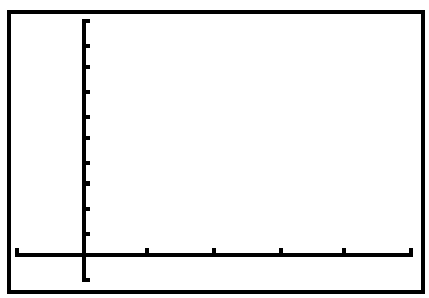
Use the viewing window shown to generate a graph of the function $f(x) = x^2$, along with the line through (1, f(1)) with a slope equal to the average rate of change between $x_1 = 1$ and $x_2 = 3$.

3. Use the graph you generated and the **intersect** feature (found by selecting **5:intersect** from the **CALC Menu**) of your graphing handheld to find the points where the line intersects the graph of f. What are the coordinates of these points?

Suppose that the rate of change of the function at a single point, for instance (1, 1), must be found. In other words, the *instantaneous rate of change* of f at $x = 1$. Right now, you have only approximated an instantaneous rate of change by computing the average rate of change between the point (1, 1) and some other point close to it.

4. Find the average rate of change of $f(x) = x^2$ between the points (1, 1) and (1.2, f(1.2)).

5. Use algebraic reasoning to find the equation of the secant line of $f(x) = x^2$ between the points (1, 1) and (1.2, f(1.2)).



6. Input this equation into **Y3** in the **Y=** editor, and graph **Y1**, **Y2**, and **Y3** in the same viewing window. Draw a sketch of the graph shown.

The instantaneous rate of change can be thought of as the slope of the line tangent to the graph at the point. In this example, the instantaneous rate of change of $f(x) = x^2$ at $x = 1$ is equal to the slope of the line tangent to the graph of f at $x = 1$.

7. Which average rate of change (your answer to Question **1** or your answer to Question **4**) is a better approximation of the instantaneous rate of change of $f(x) = x^2$ at $x = 1$? How can you get a more precise approximation? Explain.

You can approximate the instantaneous rate of change for any function f at any point $(a, f(a))$ by finding the average rate of change between $(a, f(a))$ and some other point where the function exists. As you have seen, some approximations are better than others, and the second point is thought of as being close to the first point.

If h is chosen to be a small, positive number, then the point $(a+h, f(a+h))$ is a second point on the graph of f. In addition, the point $(a+h, f(a+h))$ is to the right of the point $(a, f(a))$. The average rate of change using these two points is:

$$\frac{f(a+h) - f(a)}{h}$$

This is called the *right difference quotient*.

A point on the graph of f to the left of $(a, f(a))$ is $(a-h, f(a-h))$, so the average rate of change using these two points is:

$$\frac{f(a) - f(a-h)}{h}$$

This is called the *left difference quotient*.

A third approximation for instantaneous rate of change of f at $(a, f(a))$ is:

$$\frac{f(a+h) - f(a-h)}{2h}$$

This is called the *symmetric difference quotient*.

8. Re-examine the average rate of change that you computed for Question **4** to approximate the instantaneous rate of change at $x = 1$. Which kind of difference quotient (right, left, or symmetric) was illustrated?

9. What was used for h in the approximation described in Question **8**?

10. Re-examine the average rate of change that you computed for Question **1** to approximate the instantaneous rate of change at $x = 3$. Which kind of difference quotient (right, left, or symmetric) was illustrated? What was used for h in this approximation?

For Questions **11** through **18**, let $f(x) = x^4$.

11. Approximate the instantaneous rate of change of f at $x = 1$ by computing the right difference quotient for f at $x = 1$ with $h = 0.2$.

12. Find the left difference quotient for f at $x = 1$ with $h = 0.2$.

13. Find the symmetric difference quotient for f at $x = 1$ with $h = 0.2$.

14. Write an equation of the line through the points (0.8, 0.4096) and (1.2, 2.0736).

15. How does the your answer to Question **14** compare with your answer to Question **13**? Explain.

16. Input **X^4** into **Y1** in the **Y=** editor and the equation of the line that you found in Question **14** into **Y2**. After you have graphed these functions complete the steps below.

a. Select **4:Zdecimal** from the **ZOOM Menu**. This setting will make the distance between each row and each column of pixels equal to 0.1 units.

b. Press [TRACE], and trace along $y^1 = x^4$ until $x = 1$.

c. Select **4:SetFactors…** in the **ZOOM MEMORY Menu**, and set each zoom factor to 4.

d. Select **2:Zoom In** from the **ZOOM Menu**, and press [ENTER].

Trace the graph of f so that the crosshairs are at $x = 0.8$. What is the corresponding y-value?

17. Trace the graph of f so that the crosshairs are at $x = 1.2$. What is the corresponding y-value?

18. Find the symmetric difference quotient for f at $x = 1$ with $h = 0.0001$. What do you predict is the actual instantaneous rate of change of f at $x = 1$?

19. Let $f(x) = \sqrt{x-1}$. Approximate the instantaneous rate of change of f at $x = 1$ by computing the right difference quotient for f at $x = 1$ with $h = 0.1$.

20. Can the symmetric difference quotient be used for this function? Explain.