Proof by Mathematical Induction

Name : ______________________________

TI-Nspire™ Assessment Student 30 min

7 8 9 10 **11** 12

Question: 1.

i) Determine the sum of the first 10 cubic numbers: $1^3 + 2^3 + 3^3 + \ldots + 10^3$.

1 mark

$1^3 + 2^3 + 3^3 + \ldots 10^3 = 3025$. [1 mark]

Answer mark only. Students may use the sum command, individual entries, lists or sigma notation.

ii) Square the sum of the first 10 whole numbers and comment on the result: $(1+2+3+\ldots 10)^2$

2 marks

$(1+2+3+\ldots 10)^2 = 3025$ [1 mark]

Students should observe that the result is the same as the previous answer, but should not generalise. [1 mark]

iii) Explain how the diagram shown here relates to part (i) and (ii) above.

3 marks

Overall area, ignoring 'white spaces': (1 + 2 + 3 + 4) x (1 + 2 + 3 + 4)

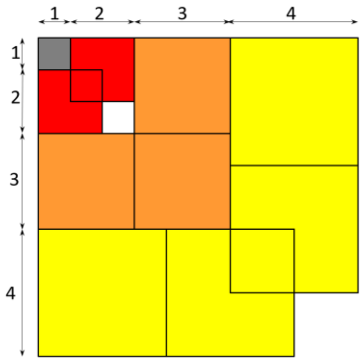
This is equal to: $(1 + 2 + 3 + 4)^2$. [Part II]

There is one 1 x 1 square, two 2 x 2 squares, three 3 x 3 and four 4 x 4.

'Overlap' fills in the white spaces.

This is equivalent to: $1 \times 1^2 + 2 \times 2^2 + 3 \times 3^2 + 4 \times 4^2 = 1^3 + 2^3 + 3^3 + 4^3$

$\therefore (1 + 2 + 3 + 4)^2 = 1^3 + 2^3 + 3^3 + 4^3$. Part (I) and (II) extend to 10.



Question: 2.

i) Express $\sum_{x=3}^{7} x^3$ in expanded form and hence evaluate the result.

2 marks

Expanded form: $3^3 + 4^3 + 5^3 + 6^3 + 7^3 = 775$. [1 mark for expanded form + 1 answer mark 775]

ii) Express: $(4+5+6+\ldots 20)^2$ using sigma $\sum$ notation and hence evaluate the result.

2 marks

$\left(\sum_{x=4}^{20} x\right)^2 = 41616$ [1 mark for sigma notation, note location of squared sign + 1 answer mark: 41616]

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Question: 3.

i) Complete the following table of values: 2 marks

n	1	2	3	4	5	6	7	8	9	10
$\sum_{x=1}^{n} x^3$	1	9	36	100	225	441	784	1296	2025	3025
$\sum_{x=1}^{n} x$	1	3	6	10	15	21	28	36	45	55
$\left(\sum_{x=1}^{n} x\right)^2$	1	9	36	100	225	441	784	1296	2025	3025

2 marks - Marks based on proportion of correct answers. Note that students can generate a table of values with the calculator making this question particularly quick for 'technology savvy' students.

ii) Determine a rule for $\sum_{x=1}^{n} x^3$, express your answer in factorised form.

2 marks

Students may use quartic regression (courtesy of the table): $\frac{x^4}{4}+\frac{x^3}{2}+\frac{x^2}{4}$ [1 mark] or prior knowledge pertaining to sums of whole numbers and information gleaned so far. Factorised form: $\frac{x^2(x+1)^2}{4}$ [1 mark]

iii) Determine a rule for $\sum_{x=1}^{n} x$, expressing the rule in factorised form.

2 marks

Students may use quadratic regression (courtesy of the table): $\frac{x^2+x}{2}=\frac{x(x+1)}{2}$

[1 mark for expanded form + 1 mark for factorised form]

iv) Use your results from part (ii) and (iii) to show that $\left(\sum_{x=1}^{n} x\right)^2 = \sum_{x=1}^{n} n^3$

2 marks

$\sum_{x=1}^{n} x \times \sum_{x=1}^{n} x = \left(\sum_{x=1}^{n} x\right)^2 = \left(\frac{x(x+1)}{2}\right)^2 = \frac{x^2(x+1)^2}{4}$ which is the same as: $\sum_{x=1}^{n} n^3$

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Question: 4.

Use mathematical induction to prove the formula for the sum of the first n^3 whole numbers.

6 marks

Show true for n = 1 LHS: $\sum_{x=1}^{n} n^3 = 1^3$ RHS: $\frac{x^2(x+1)^2}{4} = \frac{1^2 \times 2^2}{4} = 1$ [1 mark]

Assume true for n: $\sum_{x=1}^{n} n^3 = \frac{n^2(n+1)^2}{4}$ [1 mark]

Show true for n+1: LHS:

$$\sum_{x=1}^{n+1} n^3 = \sum_{x=1}^{n} n^3 + (n+1)^3$$

$$= \frac{n^2(n+1)^2}{4} + (n+1)^3$$

$$= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4}$$

[2 marks]

$$= \frac{(n+1)^2(n^2+4n+4)}{4}$$

$$= \frac{(n+1)^2(n+2)^2}{4}$$

RHS:

$$\frac{n^2(n+1)^2}{4} \quad \textit{substitute } (n+1) \textit{ for } n$$

$$= \frac{(n+1)^2(n+1+1)^2}{4}$$

[1 mark]

$$= \frac{(n+1)^2(n+2)^2}{4}$$

∴ LHS = RHS [1 mark – must include a final statement]

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