Binomial Distribution Graphs

Teacher Notes and Answers

7 8 9 10 11 **12**

TI-Nspire™ Activity Student 1.5 hours

Introduction

This is the first activity in the *Probability and Statistics with Pizzazz* series.
This activity focuses on the binomial probability mass function and its graph.
The second exploration in the series focuses on graphs of probability density functions for continuous random variables, including the normal and standard normal distributions.

A probability function may be represented as a formula (usually an equation), a table or a graph. In this activity, you will use interactive graphs in TI-Nspire CAS to explore the effect that changing the value of a parameter (a number describing some characteristic of the distribution) has on the graph of that probability function.

Exploration 1. Binomial Graphs

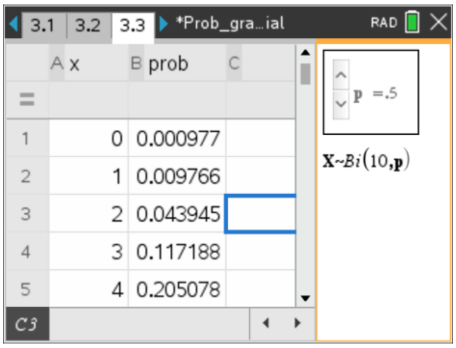
1.1 Table of the probability function for $X \sim \mathbf{Bi}(10, p)$

Consider a video game in which the player gets 10 shots at a target. Each shot is a binomial trial in which the probability of success (hitting the target) in a single trial, p, depends on the level of difficulty chosen by the player (that is, there are differing values of p).

If the random variable X is the number of successes in the 10 trials, then $X \sim \mathrm{Bi}(10, p)$ and $Pr(X = x) = {}^{n}C_{x} \cdot p^{x} \cdot (1-p)^{n-x}$, where $x = \{0,1,2,\ldots,10\}$ and $0 \le p \le 1$.

Please refer to the TI-Nspire document 'Prob_graphs_binomial'

Open the TI-Nspire document. Navigate to page 3.3. On this page, the probability of observing 0, 1, 2, …, 10 successes in 10 trials is computed in the second column of the spreadsheet. Use the slider on the right to change the value of p.



Question 1

When $p = 0.5$, describe any patterns that you observe when considering pairs of probabilities such as $\Pr(X = 0)$ and $\Pr(X = 10)$, $\Pr(X = 1)$ and $\Pr(X = 9)$. Explain why this pattern occurs.

Sample answer. Complementary pairs at opposite ends of the table have equal probabilities; for example, $\Pr(X = 0) = \Pr(X = 10) = 0.000977\ldots$ and $\Pr(X = 3) = \Pr(X = 7) = 0.11788\ldots$
This occurs because when $p = 0.5$ the probability of success is equal to the probability of failure. Consequently, the probability of observing 0 successes or 0 failures (10 successes) are equally likely. Likewise for any other complementary pair.

Question 2

For the 10 trials, when $p = 0.5$,

a. What is the 'most likely' outcome?

Answer. $\Pr(X = 5) = 0.246094\ldots$ Most likely outcome is 5 successes.

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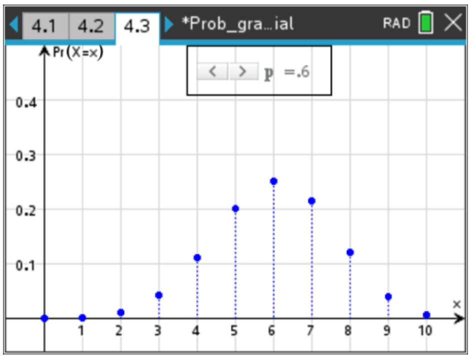
b. On average, how many successes would you expect to observe from 10 shots at the target?
Answer. Since success and failure are equally likely, on average you would expect 5 successes from 10 trials.

1.2 Graph of the probability function for $X \sim \mathrm{Bi}(10, p)$

Navigate to page 4.3 of the TI-Nspire document. The values from the probability table have been plotted in the 'Graphs' application, and the slider controls the value of p.

Consider the graphs for 10 trials and $p = 0.25$, $p = 0.5$ and $p = 0.75$ to answer Questions 3 to 5 below.



Question 3

Adjust the graph on page 4.3 for $n = 10,\ p = 0.25$.

a. Describe the shape and any other interesting features of the graph.
Sample Answer. Shape: the graph is positively skewed, showing that the number of successes on the left-hand side of the graph are most likely to be observed.

b. What is the 'most likely' number of successes from 10 trials?
Sample Answer. The 'most likely' number of successes is $x = 2$ (i.e. 2 successes and 8 failures) because this is where the maximum value of the graph occurs.

c. What are the number of successes that are unlikely to be observed?
Sample Answer. The graph indicates that 6, 7, 8, 9 or 10 successes are unlikely because the corresponding probabilities are very small (close to zero on the graph).

Question 4

Adjust the graph on page 4.3 for $n = 10,\ p = 0.5$.

a. Describe the shape and any other interesting features of the graph.
Sample Answer. Shape: the graph is symmetrical about the centre, which is at $x = 5$, showing that matching pairs of probabilities, either side of the centre, are equal.

b. What is the 'most likely' number of successes from 10 trials?
Sample Answer. The 'most likely' number of successes is $x = 5$, because this is where the maximum value of the graph occurs.

c. What are the number of successes that are unlikely to be observed?
Sample Answer. The graph indicates that 0, 1 , 9 or 10 successes are unlikely because the corresponding probabilities are very small (close to zero on the graph).

Question 5

Adjust the graph on page 4.3 for $n = 10,\ p = 0.75$.

a. Describe the shape and any other interesting features of the graph.
Sample Answer. Shape: the graph is negatively skewed, showing that the number of successes on the right-hand side of the graph are most likely to be observed. This graph is a reflection of the graph of $X \sim \mathrm{Bi}(10, 0.25)$ in the line $x = 5$.

b. What is the 'most likely' number of successes from 10 trials?
Sample Answer. The 'most likely' number of successes is $x = 8$ (i.e. 8 successes and 2 failures), because this is where the maximum value of the graph occurs. Note that this is the reverse of the case for $X \sim \mathrm{Bi}(10, 0.25)$, where 2 successes and 8 failures was most likely.

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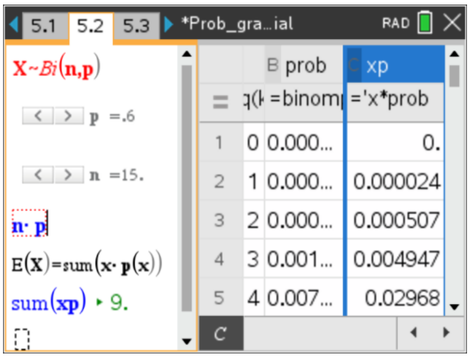
c. What are the number of successes that are unlikely to be observed?
Sample Answer. The graph indicates that 0, 1, 2, 3 or 4 **successes** are unlikely because the corresponding probabilities are very small (close to zero on the graph). Note that this is the reverse of the case for $X \sim \mathrm{Bi}(10{,}0.25)$, where 0, 1, 2, 3 or 4 **failures** are unlikely.

1.3 Expectation, $\mathrm{E}(X)$, of the binomial distribution

Navigate to page 5.2 of the TI-Nspire document. On the right window, click any cell in the spreadsheet and right-arrow to column C, labelled xp. In this column, the product of the cells a1 and b1 is calculated in cell c1 (i.e. $x \cdot prob$), and this is repeated for each value of x and its associated probability.



The definition of expectation for a discrete random variable, X, is
$\mathbf{E}(X) = \sum x \cdot \Pr(X = x)$, which can also be written $\mathrm{E}(X) = \sum x \cdot \mathrm{p}(x)$

On the left window of page 5.2, the Maths Box under the sliders calculates the product of the slider values, $n \times p$. The bottom Maths Box calculates the expected value, $\mathrm{E}(X)$, by finding the sum of column C, which is labelled xp. That is, $\mathrm{sum}(xp) = \sum x \cdot \mathrm{p}(x) = \mathrm{E}(X)$.

Question 6

On page 5.2, evaluate the Maths Boxes ‘$\mathrm{sum}(xp)$’ and ‘$n \cdot p$’: click the box and pressing <enter>.

a. Complete the table on the right by adjusting the slider values and reading the value of $\mathrm{E}(X)$ from the output of the ‘$\mathrm{sum}(xp)$’ Maths Box.

n	p	$n \times p$	$\mathrm{E}(X)$
10	0.65	6.5	6.5
20	0.25	5	5
15	0.8	12	12
12	0.35	4.2	4.2
7	0.75	5.25	5.25

b. What do you observe about the relationship between n, p and $\mathrm{E}(X)$ for these cases. Chose more values of n and p and note whether the relationship holds for any chosen values of n and p.
Sample answer. In all cases tested, $'\mathrm{sum}(xp)' = \sum x \cdot \mathrm{p}(x) = \mathrm{E}(X)$ has exactly the same value as $n \times p$. This indicates (but does not formally prove) that for any binomially distributed random variable $X \sim Bi(n, p)$, $\mathrm{E}(X) = n \times p$.

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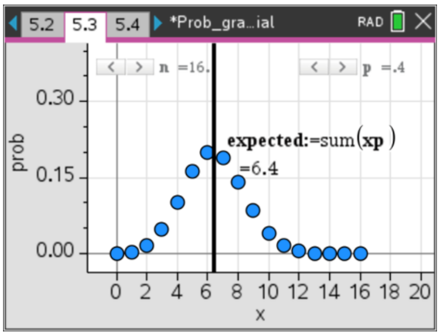
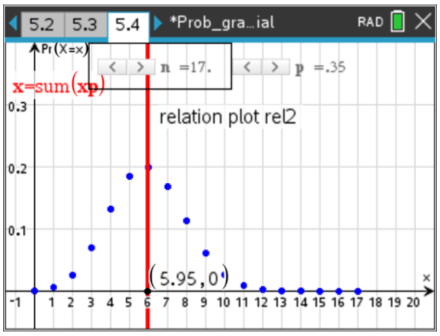
Page 5.3 displays graph of the binomial distribution in the 'Data & Statistics' application, with sliders controlling the values of n and p. The vertical line plots the expected value by calculating 'sum(xp)'. Click the vertical line to display the expected value.



Page 5.4 also displays graph of the binomial distribution in the 'Graphs' application, with sliders controlling the values of n and p. The red vertical line plots the expected value by calculating 'sum(xp)'. The expected value is shown by the x-coordinate of the intersection of the red line and the horizontal axis.



Question 5

Use the graphs to observe the any relationships between shape (skew) of the graph and the expected value of the distribution. Comment on your observations.

Shape	**Comment on location of the expected value, $\mathrm{E}(X)$**
Symmetrical	$\mathrm{E}(X)$ is at the centre of the distribution
Positively skewed	$\mathrm{E}(X)$ is at the left-hand end of the distribution, as this is where the 'most likely' observations are located.
Negatively skewed	$\mathrm{E}(X)$ is at the right-hand end of the distribution, as this is where the 'most likely' observations are located.

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