Complex Spirals

Student Activity

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TI-Nspire™ Activity Student 4 hours

Introduction

Complex numbers can be expressed in different forms. We can also create a geometric series consisting of the powers of complex numbers. When we graph the power series of complex numbers on the Argand diagram, we can observe various patterns. You will use your TI Nspire CX II CAS to investigate those patterns and establish the conditions under which they occur.

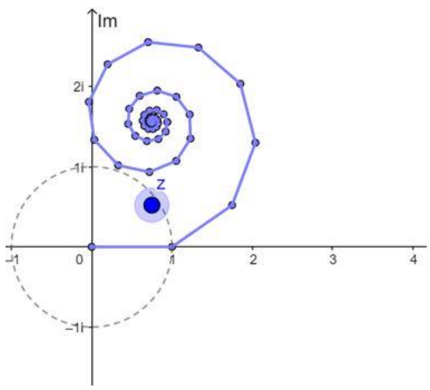
Instructions – Complex powers and complex series.

Technology Tips!

To define a complex number in a Calculator Application, select [:=].
Use the template: | □ | for modulus or use the menus.
Menu > Algebra Number > Complex Number Tools > Magnitude.
Double tap the π key to access the complex number i.

Question: 1.

For each of the following: $u = 1 + i$.



a) Find modulus $|u|$.

b) Use your TI-Nspire to find $u^0, u^1, u^2, u^3, u^5, u^6$ in Cartesian form.

c) Plot those powers on the Argand diagram.

d) What do you notice?

e) What do you think would happen if the pattern were to continue?

f) The numbers $u^0, u^1, u^2, u^3, u^5, u^6, \ldots$ follow a geometric sequence.

i) State the first term of the sequence.

ii) State the common ratio of the sequence.

Question: 2.

For each of the following: $w = \frac{1}{2} + \frac{1}{2}i$.

a) Find $|w|$.

b) Use your TI Nspire to find $w^0, w^1, w^2, w^3, w^5, w^6$ in Cartesian form.

c) Plot those powers on the Argand diagram.

d) What do you notice?

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e) What do you think would happen if the pattern were to continue?

f) Is there anything similar and/or different between the results from question 1 and question 2?

Question: 3.

Partial sums occur when we look at a geometric series $1+z+z^2+z^3+...+z^n$ and consider the following:

$S_1=1$ $\quad S_2=1+z$ $\quad S_3=1+z+z^2$ $\quad S_4=1+z+z^2+z^3$ and so on.

A partial sum can be expressed using sigma notation. For example, $S_4=\sum_{n=0}^{3} z^n$.

a) Use sigma notation on your TI Nspire to find the following sums:

(i) $\sum_{n=0}^{6} w^n$ $\quad$ (ii) $\sum_{n=0}^{8} w^n$ $\quad$ iii) $\sum_{n=0}^{10} w^n$

What do you notice?

Download the **Complex Spiral** TNS file to your Software or TI Nspire handheld.

b) Plot the first eight partial sums on the Argand diagram for $w=\frac{1}{2}+\frac{1}{2}i$ and describe the pattern.

c) Calculate: $S_{10}, S_{20}, S_{30}, S_{40}, S_{50}$.

d) What do you notice about the results from part (c)?

e) Investigate whether a similar pattern occurs for $u=1+i$

f) Repeat questions 2 & 3 for three complex numbers of your choice with the following conditions:

(i) $|z|=1$ $\quad$ (ii) $|z|>1$. $\quad$ (iii) $|z|<1$

Comment on your findings.

Question: 4.

a) Given that $1+z+z^2+z^3+...+z^n$ is a geometric series, determine the conditions when the sum of the series is convergent.

b) Find the sum to infinity for a convergent $1+z+z^2+z^3+...+z^n$ series.

c) Use your results/observations from questions 2 and 3 to verify your answer to part (b).

Question: 5.

Select a complex number in each of the four quadrants of the Argand Plane such that $|z|<1$. Investigate the behaviour of the geometric sequence $1, z, z^2, z^3 ... z^n$ and the geometric series $1+z+z^2+z^3+...+z^n$.

a) What do you notice about your results?

b) Are there any similarities and/or differences between the plots of each of the sequences and corresponding series?

c) What conclusions, if any, could you make?

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Question: 6.

For any complex number $z = x + yi$ it can be shown using DeMoivre's theorem that

$$z^n = (x+yi)^n = |z|^n \cos(n\theta) + |z|^n \sin(n\theta)i.$$

Where $|z|$ is the modulus of z, and $\theta = Arg(z)$ is the (principal) argument of z.

The spiral generated by z^n can be described by the parametric equations:

$$x = |z|^n \cos(n\theta)$$
$$y = |z|^n \sin(n\theta)$$

a) Express z^n in polar form, for:

   i. $z = 1 + i$ ii. $z = \frac{1}{2} + \frac{1}{2}i$.

b) Verify that your results in part (a) are correct by calculating z^2, z^3, z^4.

c) Using the TI-Nspire, sketch your parametric equations on a Graph application. How do your results compare with your results from questions 1 and 2?

d) Express z^n in terms of cosine and sine for each of the complex numbers that you chose in part 5, and hence sketch the corresponding spirals.

e) What do you notice about your results? Are there any similarities and/or differences?

f) What conclusions, if any, could you make?

Question: 7 (Extension)

Consider the point $z = -\frac{1}{2} + \frac{1}{2}i.$

(a) Plot the points $1, z, z^2, z^3, z^4, z^5, z^6$ on an Argand Plane

(b) Plot the spiral that is generated by z^n.

The points z, z^2, z^3 form the vertices of a triangle $\triangle ABC$.

(c) Find the side lengths and interior angles of the triangle $\triangle ABC$.

The points z^4, z^5, z^6 form the vertices of another triangle, $\triangle DEF$.

(d) Find the side lengths and interior angles of the triangle $\triangle DEF$. What do you notice?

The two triangles, $\triangle ABC$ and $\triangle DEF$ are similar.

(e) Find the scale factor, k, such that $|AB| = k|DE|$.

(f) Find the angle that you would rotate $\triangle ABC$ so that it aligns with $\triangle DEF$.

(g) Verify your results from parts (e). and (f) by looking at the triangle created by z^7, z^8, z^9.

(h) Repeat parts (a) – (f) with the point $z = \frac{2}{5} + \frac{1}{2}i.$

(i) For any complex number, $z = a + bi$ where $a, b \in \mathbb{R}$, find an expression for the scale factor and the angle of rotation that would be needed to be applied to triangle $\triangle ABC$ to obtain $\triangle DEF$.

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