Tower of Hanoi

Teachers Teaching with Technology™
Professional Development from Texas Instruments

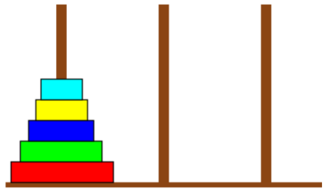
Answers

7 8 9 10 **11** 12

TI-Nspire™ Investigation Student 30 min

Introduction:

This puzzle was invented by a French Mathematician, Edouard Lucas in 1883. According to the legend, Brahmin priests moved one of the 64 disks every day according to the rules:



- Only one disk can be moved at a time;
- Disks must always be stacked on a pole;
- A large disk cannot be placed onto a smaller one.

The objective is to move all the disks from one pole to another. The legend also stated that the world would come to an end when the last disk was moved into place. If the leged is true, how long will it take to complete the puzzle? Surely if the legend was true, and the priests were still going, the world would have ended by now.

The aim of this investigation is to determine a rule that defines the number of moves to complete the puzzle consisting of a stack 'n' disks high.

Equipment:

For this activity you will need:

- TI-Nspire CX
- TI-Nspire CX file (tns): Tower of Hanoi

Teacher Notes:

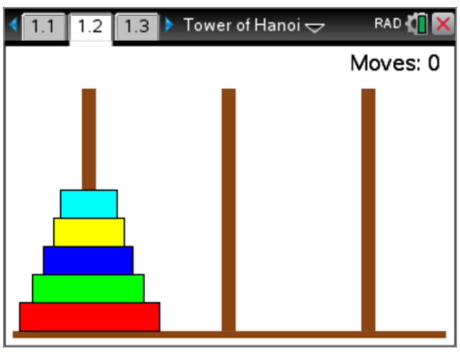
Students can use a physical models of this problem for initial exploration instead of this digital resource, however the act of counting and tracking moves can distract from the observation of pattern. The digital model allows students to focus on the pattern whilst accurately recording the number of moves. Feedback is also provided with regards to whether the problem could be solved in fewer moves.

Instructions

Open the file:

Tower of Hanoi

Navigate to the problem page (shown opposite).



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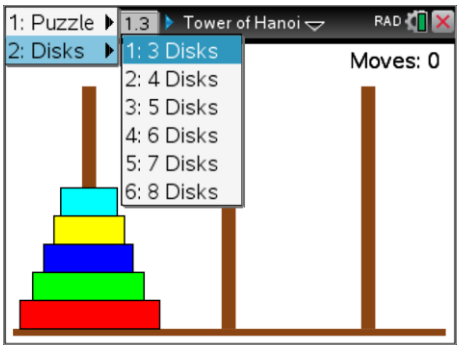
Setting the number of disks

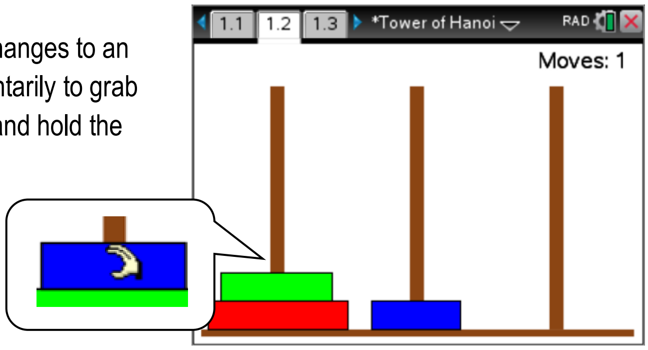


Use the **[menu]** key and set the number of disks to three.

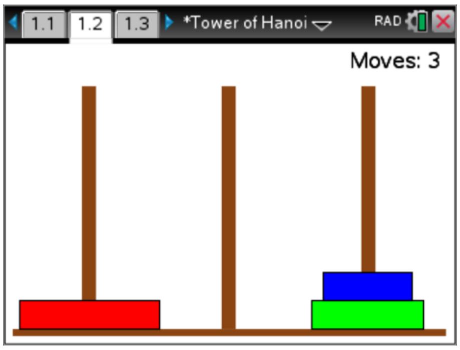
✍ The single and two disk problems will be determined by reasoning rather than modelling.

Moving Disks

Move the mouse over the top disk, the mouse changes to an open hand. Press and hold the touchpad momentarily to grab the disk. Move the disk to a new column, press and hold the touchpad momentarily to release the disk.



Moves

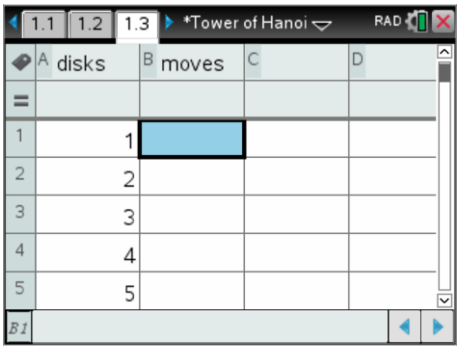


The number of moves is automatically recorded in the top right corner of the screen.

✍ Once a move has been made, you can't take it back.

Solve the three disk puzzle and make a note of the number of moves.

Entering Data



Navigate to the spreadsheet on page 1.3

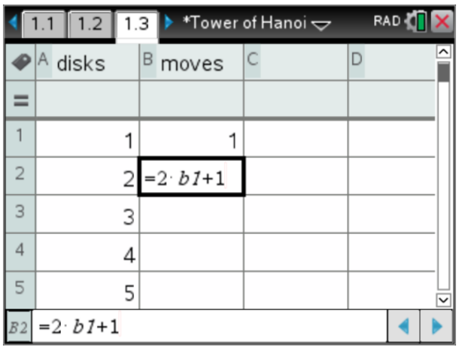
Enter the numbers 1 to 8 in column A.

Leave the first two entries in column B blank. These will be included by reasoning and logic later. Record the minimum number of moves used to solve the three disk problem in cell B3.

Teacher Notes

Students use a spreadsheet to tabulate results. If students have a reasonable working knowledge of spreadsheets they should be encouraged to develop a recursive formula for the rule relating the number of disks and the number of moves required to solve this problem.

Note that standard spreadsheet notation works in TI-Nspire. Relative and absolute ($) cell referencing work as expected.



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Questions

1. Use the puzzle to determine and record the minimum number of moves required to solve:
   a. The three disk problem. 7 moves
   b. The four disk problem. 15 moves

If the number of moves increases by the same amount each time a disk is added the puzzle, the relationship is said to be linear.

2. If the relationship between disks and moves is linear, how many moves should it take to solve the five disk problem?
   Three disk problem = 7. Four disk problem = 15. Increased by 8 moves. If the relationship is linear then the five disk problem will take 23 moves. (15 + 8).

   Teacher Notes:

   This question combined with the linearity definition (above) is aimed at focusing students on number patterns. Students should be able to recognise linear and non-linear number patterns, not just their visual and graphical representation.

3. Use the puzzle to determine the minimum number of moves required to solve the five disk problem and hence determine if the relationship is linear or non-linear.
   Record your answer to the five disk problem in the spreadsheet.
   The relationship is not linear. The minimum number of moves for the 5 disk problem is 31. The first difference was 8, the second difference was 16. The relationship is therefore non-linear.

4. Determine the number of moves required to solve the one and two disk problems. Explain how you determined these values.
   Record the number of moves in the spreadsheet.
   One disk problem = 1 move. Two disk problem = 3 moves. Students may have determined these experimentally or used the number pattern to establish the results; their responses to this question should indicate which approach they used. Note that this question provides further evidence that the model is not linear.

5. Navigate to the graph page, with your data for the one, two ... and five disk problems in the spreadsheet you will see five points on the graph. Do the points on the graph form a straight line or a curve?
   Curve (visual evidence the solution is not linear).

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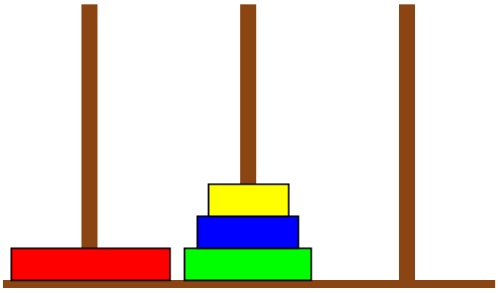
6. The image shown opposite is a partial solution to the four disk problem. The large disk has not yet been moved.



   a. How many moves have taken place so far?
      The three disk problem has essentially been solved therefore 7 moves have taken place..

      Teacher Notes:

      This question aims to scaffold student thinking towards the recursive relationship to double the previous amount and add one (move last disk).

   b. Without referring to your previous data, how many moves are required to complete this puzzle?
      One more move to move the large disk plus 7 more moves to re-stack the smaller ones.

   c. Explain how you determined your answer to the previous question.
      Double the number of moves for the three disk problem plus one move for the bottom disk.

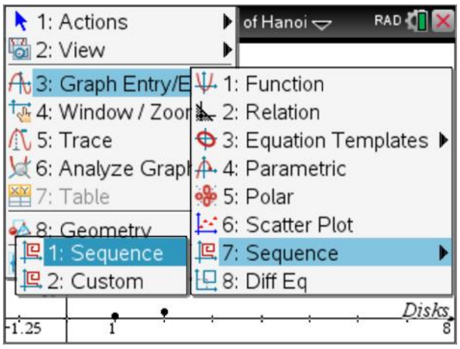
      Teacher Notes:

      The aim of this question is to see if students are able to articulate the formula and describe the recursive relationship as doubling the previous amount then adding one.

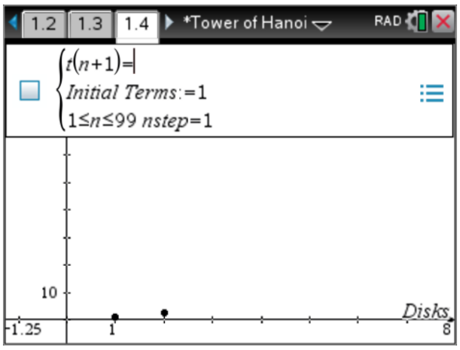
Developing a Rule

Navigate to the graph on page 1.4

Change the graph type to: **Sequence**



Change the sequence formula to your preferred notation. In the screen shown opposite u(n) has been changed to t(n+1).



- Enter **your** formula using the appropriate notation. With the subject of the equation set to t(n+1), the previous term can be referenced as t(n).
- The next important component of the formula is to set the initial conditions. If the puzzle consisting of zero disks would require zero moves, therefore t(0) = 0.
  This would align to the *initial term* (moves) equal to zero and the *nstep* (n) also equal to zero.
  An alternative is to consider the puzzle to start with just one disk that requires just one move to solve. In this case: *nstep* = 1 and the initial term would be 1.
  Each of these entries can be edited by navigating to the corresponding value replacing it.

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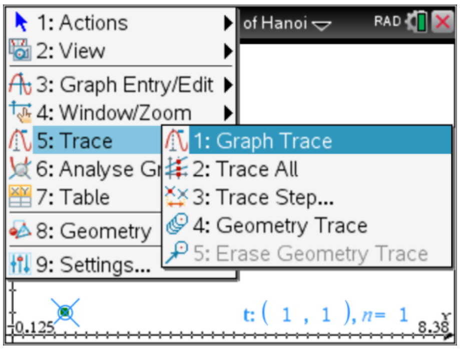
You can visually check your rule using the graph and numerically using the TRACE option. The graph being traced is indicated by the expression in the bottom right corner of the screen. "S1" indicates the scatter plot using the data from the spreadsheet, "t" indicates the sequence.



Another option to check the values for each point is to press Ctrl + T (Table) to toggle a table of values on/off.

Alternatively, navigate to page 1.5 (Calculator application) and type: t(3) then press [Enter] to see the number of moves required to solve the three disk problem.

7. The legend states that the puzzle consisted of 64 disks. Use your formula to determine the number of days (and years) that would be required to solve the puzzle.
The number is too large for the calculator to compute and display all the digits, based on the digits computed, the answer is: 1.844674407371 x 10^{19} or 18446744073710 million moves. Even if a move was executed every day and started at the dawn of earth's first sunrise, the puzzle would still be a long way from completion.

8. Another way to explore the rule for the total number of moves is to examine the number of moves for each disk in the puzzle. Complete the table below to help record the number of moves for each disk. Formulate a rule for the total number of moves using this method. (The one and two disk puzzles have already been completed)

Disk:	One	Two	Three	Four	Five	Six	Total
One disk puzzle	1						1
Two disk puzzle	2	1					3
Three disk puzzle	4	2	1				7
Four disk puzzle	8	4	2	1			15
Five disk puzzle	16	8	4	2	1		31
Six disk puzzle	32	16	8	4	2	1	63

Students may realise that these totals are also the same as $2^n - 1$ or as: $\sum_{n=0}^{x-1} 2^n$ so the recursive formula $t(n+1) = 2t(n) + 1$ is equivalent to both of these alternative calculations.