Tetrahedral Numbers

Teacher Notes & Answers

7 8 9 10 11 **12**

TI-30XPlus MathPrint™ | Activity | Student | 50 min

Teacher Notes:

This activity is an extension of the Triangular Numbers activity. The triangular numbers appear in Pascal's triangle, so too the Tetrahedral numbers. From there it is also possible to generate a formula for the sum of the first n^2 numbers which leads into a whole other dimension of possibilities. Once all the formulas have been established via alternative means, they can be proved by induction! Use the PowerPoint slides to help students visualise.

Finding Patterns

What are the Tetrahedral numbers? The prefix 'tetra' refers to the quantity four, so it is not surprising that a tetrahedron consists of four faces, each face is a triangle. This triangular formation can sometimes be found in stacks of objects. The series of diagrams below shows the progression from one layer to the next for a stack of spheres.

Row Number	1	2	3	4	5
Items Added					
Complete Stack					

Question: 1.

Create a table of values for the row number and the corresponding quantity of items that are added to the stack.

Answer:

Row	1	2	3	4	5
Items:	1	3	6	10	15

Question: 2.

Create a table of values for the row number and the corresponding quantity of items in a complete stack.

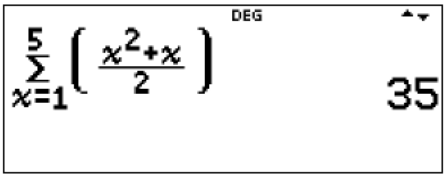
Answer:

Row	1	2	3	4	5
Items:	1	4	10	20	35

Author: P. Fox

Question: 3.

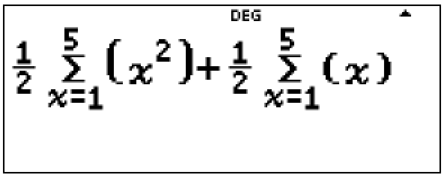
The calculator screen shot shown here illustrates how to determine the fifth tetrahedral number. The same command could be used to determine any of the tetrahedral numbers.
Explain how this command is working.



Answer: The sum of first *n* whole numbers = n x (n + 1) ÷ 2 (triangular numbers) is the expression in the summation (sigma) command. From the diagrams, it can be seen that each time another row is 'added' to the stack, to create the next tetrahedral number, the quantity to be added is a triangular number.

Question: 4.

Verify that the calculation shown opposite is the same as the one generated in Question 3.



Answer: Students may use a single calculation, however they should note the general case. This involves an extraction of the common factor (1/2) [Application of the distributive law], furthermore, the original expression: $(x^2 + x)$ has simply been split into two separate calculations. [Application of the commutative law for addition.]

Question: 5.

Enter the numbers 1, 2 … 10 in List 1 on the calculator. Enter the first 10 tetrahedral numbers in List 2. Once the values have been entered try the following:

a) Quadratic regression using List 1 and List 2. Check the validity of the result via substitution.

Answer: As the relationship is not quadratic, the quadratic regression does not predict the values for the tetrahedral numbers. Equation: $3.25x^2 - 12.35x + 14.3$. Students should see instantly that this equation is not going to predict values correctly. Substituting x = 0, x = 1 … produce erroneous results.

b) Cubic regression using List 1 and List 2. Check the validity of the result via substitution.

Answer: Cubic regression seems to do a perfect job of a formula:

Equation: $\frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x$ which can also be represented as: $\frac{x^3+3x^2+2x}{6} = \frac{x(x+1)(x+2)}{6}$. Students can use direct substitution or define a function on the calculator and see that f(1)=1, f(2)=4, f(3)=10 … f(10)=220.

Pascal's Triangle – Hidden Gem

Pascal's triangle also contains the tetrahedral numbers.

Notice: The *n*th triangular number is in row[1] *n*+2.

Example: The number 20 is the 4th triangular number and it is located in the 6th row.

Recall that the elements in Pascal's triangle can be computed using combinatorics: ${}^{n}C_{r} = \frac{n!}{(n-r)!r!}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

[1] Row numbering in Pascal's triangle starts at row(0) = {1}, row(1) = {1, 1}, row(2) = {1, 2, 1}

Author: P. Fox

Question: 6.

Use combinatorics to determine the value of the 100th Tetrahedral number. Check your answer using the cubic equation established in Question 5 and the summation tool on the calculator.

Answer:

Using combinatorics: ${}^{102}C_3 = 171700$.

Using the cubic equation from Question 5: $\frac{x(x+1)(x+2)}{6} = \frac{100\times101\times102}{6} = 171700$

Using the calculator's summation command: $\sum_{x=1}^{100} \frac{x(x+1)}{2} = 171700$

Question: 7.

Use Pascal's triangle to determine a formula for the Tetrahedral numbers.

Answer:
$$\begin{aligned} {}^{n+2}C_3 &= \frac{(n+2)(n+1)n(n-1)(n-2)...}{3!(n-1)(n-2)...} \\ &= \frac{n(n+1)(n+2)}{6} \end{aligned}$$
[Same equation established using cubic regression]

Question: 8.

Given that the Tetrahedral numbers can be computed using:

$$\sum_{n=1}^{x}\left(\frac{n^2+n}{2}\right)$$

Use induction to show that this is equal to $\frac{x(x+1)(x+2)}{6}$.

Answer

Step 1: Show true for $x = 1$

We must first prove that the formula is true for $x = 1$.

RHS: $\frac{x(x+1)(x+2)}{6} = \frac{1\times2\times3}{6} = 1$

LHS: $\sum_{n=1}^{1}\left(\frac{n^2+n}{2}\right) = 1$. Therefore LHS = RHS

Step 2: Assume true for x

That is: $\sum_{n=1}^{x}\left(\frac{n^2+n}{2}\right) = \frac{x(x+1)(x+2)}{6}$ **-- Equation 1**

Author: P. Fox

Step 3: Show true for $x + 1$.

Working with the LHS

LHS = $\sum_{n=1}^{x+1}\left(\frac{n^2+n}{2}\right)=\sum_{n=1}^{x}\left(\frac{n^2+n}{2}\right)+\frac{(x+1)^2+(x+1)}{2}$

From Equation 1 we can re-write this as: $\frac{x(x+1)(x+2)}{6}+\frac{(x+1)^2+(x+1)}{2}$

Common denominator: $\frac{x(x+1)(x+2)}{6}+\frac{3(x+1)^2+3(x+1)}{6}$

Common factor: $\frac{(x+1)\left[x(x+2)+3(x+1)+3\right]}{6}$

This simplifies to: $\frac{(x+1)(x^2+5x+6)}{6}$

Factorising: $\frac{(x+1)(x+2)(x+3)}{6}$

Working with the RHS:

Replace x with $x+1$ so RHS becomes: $\frac{(x+1)(x+2)(x+3)}{6}$

$\therefore$ LHS = RHS

Question: 9.

Question 4 used the property that $\sum_{n=1}^{x}\frac{n^2+n}{2}=\frac{1}{2}\sum_{n=1}^{x}n^2+\frac{1}{2}\sum_{n=1}^{x}n$.

Use this to show that: $\sum_{n=1}^{x}n^2=\frac{x(x+1)(x+2)}{6}$

Answer

$$\begin{aligned}
\sum_{n=1}^{x}n^2 &= 2\sum_{n=1}^{x}\frac{n^2+n}{2}-\sum_{n=1}^{x}n \\
&= \frac{x(x+1)(x+2)}{3}-\frac{x^2+x}{2} \\
&= \frac{2x^3+6x^2+4x}{6}-\frac{3x^2+3x}{6} \\
&= \frac{2x^3+3x^2+x}{6} \\
&= \frac{x(x+1)(2x+1)}{6}
\end{aligned}$$

Author: P. Fox

Question: 10.

Use induction to prove that: $\sum_{n=1}^{x} n^2 = \frac{x(x+1)(2x+1)}{6}$.

Answer

Step 1: Show true for $x = 1$

LHS: $\sum_{n=1}^{1}\left(n^2\right)=1$

RHS: $\frac{1\times(1+1)\times(2+1)}{6}=1$. $\therefore$ LHS = RHS

Step 2: Assume true for x

That is: $\sum_{n=1}^{x}\left(n^2\right)=\frac{x(x+1)(2x+1)}{6}$ **-- Equation 1**

Step 3: Show true for x + 1.

Working with the LHS

$$\sum_{n=1}^{x+1}\left(n^2\right)=\sum_{n=1}^{x}\left(n^2\right)+(x+1)^2$$

$$\sum_{n=1}^{x+1}\left(n^2\right)=\frac{x(x+1)(2x+1)}{6}+(x+1)^2$$ -- From Equation 1:

$$\sum_{n=1}^{x+1}\left(n^2\right)=\frac{x(x+1)(2x+1)}{6}+\frac{6(x+1)^2}{6}$$

$$=\frac{(x+1)(2x^2+x+6x+6)}{6}$$

$$=\frac{(x+1)(x+2)(2x+3)}{6}$$

Working with the RHS

$\frac{x(x+1)(2x+1)}{6}$ replace x with $x+1$

$\frac{(x+1)(x+2)(2x+3)}{6}$ $\therefore$ LHS = RHS

Author: P. Fox